1. By passing to projectivizations, it is easy to see that all $m$-dimensional (projective) linear subspaces $L \subset \mathbb{P}^n$ are parametrized by the Grassmannian $Gr(m+1, n+1)$. Now let $X \subset \mathbb{P}^n$ be a hypersurface defined by a degree $d$ homogeneous polynomial $H(x_0, \ldots, x_n)$, and let $Z \subset Gr(m+1, n+1)$ be the subset corresponding to those linear subspaces $L \subset X \subset \mathbb{P}^n$ which actually belong to $X$. Show that $Z$ is a closed subset of the Grassmannian. (HINT: Let $W \subset k^{n+1}$ the linear subspace in $k^{n+1}$ corresponding to $L \subset \mathbb{P}^n$. Then $L \subset X$ iff the restriction of $H$ to $W$ is identically zero. Denote by $S$ the universal subbundle on $Gr = Gr(m+1, n+1)$. Show that $Z \subset Gr(m+1, n+1)$ is the set of points where a certain section $s \in H^0(Gr, (Sym^d(S))^*)$ vanishes (the star stands for the dual vector bundle). Conclude that $Z$ is closed.)

2. Normally one only defines Hilbert schemes for projective varieties (this is because of the way the Hilbert polynomial is defined). However, when the Hilbert polynomial is constant, i.e. $P(d) = m$, one can consider the Hilbert scheme of all length $m$ zero-dimensional subschemes $\xi \subset X$ for any $X$. More precisely, such subschemes are given by quotients $O_X \to O_\xi \to 0$ where $O_\xi$ is supported at finitely many closed points of $X$ and $\dim_k H^0(X, O_\xi) = m$.

Let $Hilb^m(X)$ be the Hilbert scheme parametrizing all such quotients (it is not hard to prove that if $X \subset Y$ with $Y$ projective and $X$ open in $Y$, then $Hilb^m(X)$ is an open subscheme of $Hilb^m(Y)$). The purpose of this exercise is to identify $Hilb^m(k^2)$. Since $X = k^2$ is affine with the ring of regular functions $A = k[x, y]$, every sheaf of the type $O_\xi$ comes from $A$-module $A/I$ where $I \subset A$ is an ideal. Consider the support of $\overline{A/I}$, i.e. the set of all prime ideals $p \subset A$ such that $(A/I)_p \neq 0$.

(a) Show that the support coincides with the set $V(I)$ of all prime ideals which contain $I$.

(b) Suppose that $A/I$ is finite dimensional over $k$. Prove that every ideal in $V(I)$ is maximal.

(c) Suppose that $A/I$ is infinite-dimensional over $k$. Show that there exists $P \in V(I)$ such that $A/P$ is infinite dimensional. (HINT: use the filtration $M_1 \subset M_2 \subset \ldots \subset M_n = A/I$ with $M_i / M_{i-1} \simeq A/P_i$ and part (a) above). Show, in addition, that such $P$ cannot be maximal.
Parts (b), (c) mean that the sheaf $\widetilde{A/I}$ corresponds to a zero-dimensional subscheme iff $A/I$ is finite dimensional over $k$. Hence $\text{Hilb}^m(k^2)$ simply parametrizes all ideals in $A$ of finite codimension.

(d) Let $I \subset k[x, y]$ be an ideal of finite codimension $m$. Fix an $m$-dimensional vector space $V$ over $k$ and a vector space isomorphism $A/I \simeq V$. Let $v \in V$ be the image of $1 \in A$, and $B_1, B_2$ be the two linear operators on $V$ which correspond to the action of $x$ and $y$ on $A/I$, respectively. Prove that

The operators $B_1$, $B_2$ commute and any vector subspace $W \subset V$ which contains $v$ and satisfies $B_1(V) \subset V$, $B_2(V) \subset V$, is necessarily equal to $V$ itself.

The triples $(v, B_1, B_2)$ having this property are called stable.

(e) Show that the set of all stable triples $U$ is an open subset of $V \times C$ where $C \subset \text{End}(V) \times \text{End}(V)$ is the closed subset formed by pairs of commuting operators.

(f) Show that $\text{Hilb}^m(k^2) \simeq U/GL(V)$ where an element $g \in GL(V)$ acts on $U$ by taking $(v, B_1, B_2)$ to $(gv, gB_1g^{-1}, gB_2g^{-1})$. Show, in addition, that the $GL(V)$-action on $U$ is free.

The last result may be used to show that $\text{Hilb}^m(k^2)$ is smooth and irreducible of dimension $2m$. 

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