THE GENERATION OF GRAVITATIONAL WAVES:

A REVIEW OF COMPUTATIONAL TECNIQUES *†

Kip S. Thorne

California Institute of Technology

Pasadena, California

TABLE OF CONTENTS

*Sections 1-6 of these lectures were presented at the International School of Cosmology and Gravitation, Erice, Sicily, 13-25 March 1975. Section 7, which is included here for completeness, was presented at the Second Latin-American Symposium on Relativity and Gravitation, Caracas, Venezuela, December 8-13, 1975. These are the lecture notes passed out at those conferences, except for a modest (and inadequate!) amount of updating as of January 1977.

†Supported in part by the National Aeronautics and Space Administration (NGR 05-002-256) and the National Science Foundation (AST 75-01398 A01). The updating and finalizing of the manuscript was carried out at Cornell University with partial support from the National Science Foundation (AST 75-21153).

1. INTRODUCTION

1.1 Nature of these lectures

In these lectures I shall review techniques for calculating gravitational-wave generation. My emphasis will be on the techniques themselves, on their realms of validity, and on typical applications of them. Most derivations will be omitted or sketched only briefly, but I shall give references to places where the derivations can be found.

I shall presume that the reader is familiar with general relativity at the level of "track one" of Misner, Thorne, and Wheeler (1973) -- cited henceforth as "MTW." My sign conventions and notation will be the same as in MTW.

1.2 Regions of Spacetime Around a Source

One can characterize a source of gravitational waves, semi-quantitatively, by the following length scales:

$$L \equiv \text{"Size of source"} \equiv \left\{ \begin{array}{l} \text{radius of region inside which} \\ \text{the stress-energy } T^{\alpha\beta} \text{ is contained} \end{array} \right\},$$

$$2M \equiv \text{"Gravitational radius of source"} \equiv \left\{ \begin{array}{l} 2 \times \text{mass of source in} \\ \text{units where } G = c = 1 \end{array} \right\},$$

$$\lambda\!\!\!^{-} \equiv \text{"reduced wavelength of waves"} \equiv \left\{ \begin{array}{l} 1/2\pi \times \text{characteristic wave-} \\ \text{length of gravitational} \\ \text{waves emitted by source} \end{array} \right\}, \quad (1.2.1)$$

$$\left. \begin{array}{l} r_I \equiv \text{"inner radius of local wave zone"} \\ r_0 \equiv \text{"outer radius of local wave zone"} \end{array} \right\} \text{ (see below).}$$

Corresponding to these length scales, one can divide space around a source into the following regions (See Fig. 1):

<u>Source</u>: $r \equiv \text{radius} \lesssim L$

<u>Strong-field region</u>: $r \lesssim 10\ M$ if $10\ M \gtrsim L$
typically does not exist if $L >> 10\ M$

<u>Weak-field near zone</u>: $L<r,\ 10\ M<<r,\ r<<\lambda\!\!\!^{-}$ (1.2.2)

<u>Wave generation region</u>: $r \lesssim r_I$ (includes source, strong-field region, and weak-field near zone)

<u>Local wave zone</u>: $r_I \lesssim r \lesssim r_0$

<u>Distant wave zone</u>: $r_0 \lesssim r$.

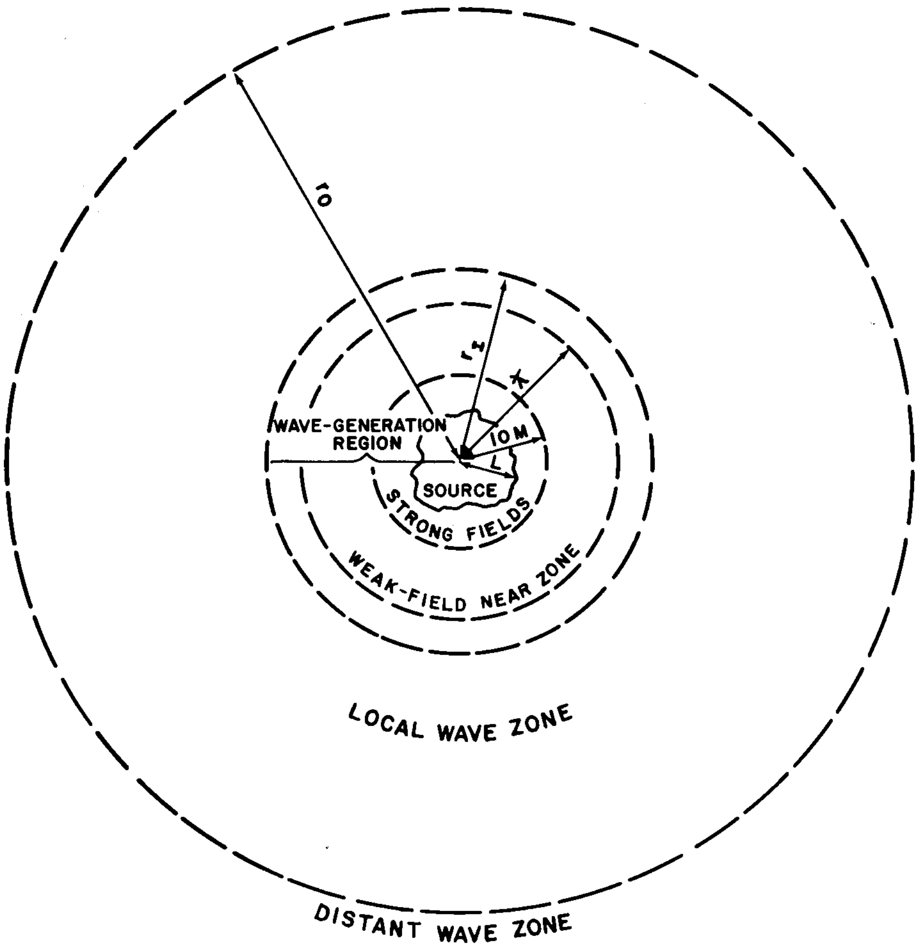

Figure 1. Regions of spacetime surrounding a source.

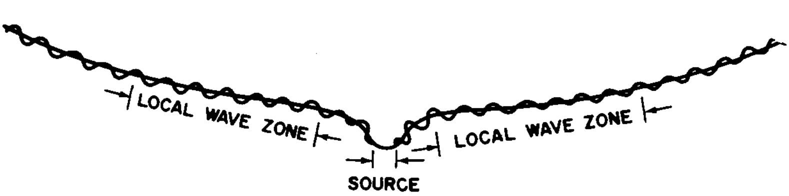

Figure 2. The local wave zone (schematic diagram). The smooth curve depicts the curvature of the background spacetime on which the waves (rippled curve) propagate. Near the source one cannot necessarily split the spacetime geometry into background plus waves; but in the local wave zone one can.

The "local wave zone" is the region in which (i) the source's waves are weak, outgoing ripples on a background spacetime; and (ii) the effects of the background curvature on the wave propagation are totally negligible (see Fig. 2).

The inner edge of the local wave zone r_I is the location at which one or more of the following effects becomes important: (i) the waves cease to be waves and become near-zone field, i.e., r becomes $\lesssim ƛ$; (ii) the gravitational pull of the source produces a significant redshift, i.e., r becomes $\sim$ 2M = (Schwarzschild radius of source); (iii) the background curvature produced by the source distorts the wave fronts and backscatters the waves, i.e., $(r^3/M)^{1/2}$ becomes $\lesssim ƛ$; (iv) the outer limits of the source itself are encountered, i.e., r becomes $\lesssim$ L = (size of source). Thus, the inner edge of the local wave zone is given by

$$r_I = \alpha \times \text{Maximum } \{ƛ,\ 2M,\ (Mƛ^2)^{1/3},\ L\}\ , \tag{1.2.3}$$

$$\alpha \equiv \text{(some suitable number large compared to unity).}$$

The outer edge of the local wave zone r_O is the location at which one or more of the following effects becomes important: (i) a significant phase shift has been produced by the "M/r" gravitational field of the source, i.e., $(M/ƛ) \cdot \ell n(r/r_I)$ is no longer $<< \pi$; (ii) the background curvature due to nearby masses or due to the external universe perturbs the propagation of the waves, i.e., r is no longer $<< R_B$ = (background radius of curvature). Thus, the outer edge of the local wave zone is given by

$$r_O = \text{Minimum } \{r_I \cdot \exp(ƛ/\beta M),\ R_B/\gamma\}, \tag{1.2.4}$$

$$\beta,\gamma \equiv \text{(some suitable numbers large compared to unity).}$$

Of course, we require that our large numbers α, β, γ, be adjusted so that the thickness of the local wave zone is very large compared to the reduced wavelength,

$$r_O - r_I >> ƛ. \tag{1.2.5}$$

Throughout these lectures I shall confine attention to sources which possess a local wave zone -- and I shall call such sources "isolated." It seems likely that every source of gravitational waves in the Universe today is "isolated." However, in the very early Universe the background curvature, $1/R_B^2$, was so large that sources might not have been isolated.

In complex situations the location of the local wave zone might not be obvious. Consider, for example, a neutron star passing very near a super-massive black hole. The tidal pull of the hole sets the neutron star into oscillation, and the star's oscillations produce gravitational waves (Mashoon 1973; Turner 1977). If the hole is large enough, or if the star is far enough from it, there may exist a local wave zone around the star which does not also enclose the entire hole. Of greater interest - because more radiation will be produced - is the case where the star is very near the hole and the hole is small enough ($M_h \lesssim 100 M_\odot$) to produce large-amplitude oscillations, and perhaps even disrupt the star. In this case, before the waves can escape the influence of the star, they get perturbed by the background curvature of the hole. One must then consider the entire star-hole system as the source, and construct a local wave zone that surrounds them both.

The local wave zone acts as a buffer between the wave-generation region $r < r_I$ and the distant wave zone $r > r_O$. The existence of this buffer enables one to separate cleanly the theory of wave generation (applicable for $r < r_O$; treated in §§ 2-6 of these lectures) from the theory of wave propagation (applicable for $r > r_I$; treated in § 7 of these lectures).

1.3 The Gravitational-Wave Field

In the local wave zone where gravity is weak we shall use coordinate systems $(t,x,y,z) \equiv (x^0,x^1,x^2,x^3)$ which are centered on the source and are very nearly Minkowskiian; and we shall sometimes introduce the corresponding spherical coordinates (t,r,θ,ϕ) with

$$x = r \sin\theta \cos\phi \ , \ y = r \sin\theta \sin\phi \ , \quad z = r \cos\theta. \qquad (1.3.1)$$

The components of the metric then differ only slightly from Minkowskii form

$$g_{\alpha\beta} \equiv \eta_{\alpha\beta} + h_{\alpha\beta}, \ \eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1), \ |h_{\alpha\beta}| << 1. \qquad (1.3.2)$$

Throughout these lectures it will be adequate, in the local wave zone, to treat $h_{\alpha\beta}$ as a linearized field residing in flat spacetime. If one knows $h_{\alpha\beta}$ in any such coordinate system (i.e. in any "gauge"), one can compute from it the "gravitational wave field" (§35.4 of MTW)

$$h^{TT}_{jk} \equiv (P_{ja} h_{ab} P_{bk} - 1/2 \ P_{jk} P_{ab} h_{ab}). \qquad (1.3.3)$$

Here Latin indices run from 1 to 3; repeated Latin indices must be summed even if they are both "down"; and "TT" means "transverse, traceless part". The projection tensor used in this computation is

$$P_{jk} \equiv \delta_{jk} - n_j n_k \ \ ; \ n_j \equiv x^j / r. \qquad (1.3.4)$$

The task of wave-generation theory (§§ 2-6) is to determine h^{TT}_{jk} in the local wave zone. Once that has been done, wave-propagation theory (§ 7) can be used to calculate h^{TT}_{jk} in the distant wave zone.

1.4. Formalisms for Calculating Wave Generation

To calculate the generation of gravitational waves one must solve simultaneously the Einstein field equations and the equations of motion of the source. These equations are so complicated that one cannot hope to solve them exactly in any realistic situation. Therefore, one must resort to approximation schemes.

Several different approximation schemes have been devised for handling different types of sources. Most of these approximation schemes have been written in the form of "plug-in-and-grind" formalisms (i.e., computational algorithms). In these lectures I shall describe the following formalisms:

Weak-field formalisms (§2). These formalisms are valid for sources with weak internal gravitational fields ($L \gg 2M$; sources without any strong-field region; cf. Fig. 1). Section 2.1 will present a classification of weak-field formalisms and their realms of validity; and §§2.2-2.7 will present specific weak-field formalisms, each with its own realm of validity. A catalogue of the weak-field formalisms is given in Table 1.

Multipole analysis of the radiation field (§3). The gravitational waves from any source can be resolved into multipole fields, and that multipole resolution yields simple formulas for the energy, linear momentum, and angular momentum carried off by the waves. Section 3 presents such a multipole analysis, restricted to the local wave zone. That analysis can be used fruitfully in conjunction with the wave-generation formulas of §§2,4, and 5.

Slow-motion formalisms (§4). These formalisms are valid for sources with slow internal motions ($L/\lambda\!\!\!^{-} \ll 1$). They make use of near-zone multipole-moment expansions, and consequently they tie in tightly with the multipole analysis of the radiation field (§3). The slow-motion formalisms are catalogued in Table 1 along with the weak-field formalisms.

Small-perturbation formalisms (§5). These formalisms are applicable to sources consisting of small dynamical perturbations of a nonradiative "background" system (e.g., small particles falling into black holes, and small-amplitude pulsations of stars).

TABLE 1

A CATALOGUE OF WEAK-FIELD FORMALISMS AND SLOW-MOTION FORMALISMS

← WEAK-FIELD EXPANSION (downward) — SLOW-MOTION EXPANSION →

Slow-motion expansion parameter: $\left(\delta h^{TT}_{jk}\right)_{\ell\text{-pole}} \Big/ \left(h^{TT}_{jk}\right)_{\ell\text{-pole}}$

$\left(\frac{\delta T^{\alpha\beta}}{T^{00}}\right.$	$\left.\frac{\delta \bar{h}^{\alpha\beta}}{\bar{h}^{00}}\right)$	$\left(\frac{L}{\lambda\!\!\!^{-}}\right)^2$	$\left(\frac{L}{\lambda\!\!\!^{-}}\right)^4$	Complete Accuracy
ε	1			Special Relativity (§2.2)
ε	ε			Linearized Theory (§2.3)
ε^2	ε	Newtonian theory (§2.6)		Post-Linear Theory (§2.4)
ε^2	ε^2	Quadrupole-Moment Formalism (§2.6); also, weak-field limit of Slow-Motion Formalism (§4.4)		Post-Linear Wave-Generation Formalism (§2.5)
ε^3	ε^2			
ε^3	ε^3		Post-Newtonian Wave-Generation Formalism (§§2.7 and 4.5)	
Complete Accuracy		Slow Motion, Multipole-Moment Formalism (§4.3); errors $M/\lambda\!\!\!^{-}$		

Notation:

δ	means "errors in."
$\bar{h}^{\alpha\beta}$	is the gravitational field (including both nonradiative and radiative parts)
h^{TT}_{jk}	is the radiation field.
ε	is the magnitude of the source's internal gravitational fields.
L	is the source's linear size.
M	is the source's mass.
$\lambda\!\!\!^{-}$	is the reduced wavelength of the gravitational waves.

Formalisms for studying systems with strong internal fields and fast, large-amplitude internal motions (§6). Unfortunately, no analytic formalisms now exist for studying such systems. However one can study their evolution and the waves they emit by numerical solution of the Einstein field equations on a large computer. The necessary numerical techniques are now under development.

2. WEAK-FIELD FORMALISMS

2.1 Classification of Weak-Field Formalisms

Weak-field formalisms are applicable to systems with weak internal gravitational fields.

To compute a characteristic dimensionless strength ε of the internal field of a source, one can analyze the source as though spacetime were flat, using globally Minkowskii coordinates in which the center-of-mass is at rest. In these coordinates one can compute ε from the retarded integral

$$\varepsilon = \underset{\substack{\text{Maximum over all}\\ \text{"relevant" values}\\ \text{of the field point}\\ (t,x^j)}}{} \left\{ \int \frac{T^{00}(t-|\underset{\sim}{x} - \underset{\sim}{x}'|, \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d^3x' \right\} . \tag{2.1.1}$$

Here T^{00} is the time-time component of the stress-energy tensor, and the "relevant" field points are those points at which one portion of the source interacts with fields produced by other portions of the source. One need not know ε with high precision; errors as large as a factor 3, say, are perfectly allowable.

For a source consisting of a single coherent body (e.g., a pulsating star) with mass M and linear size L,

$$\varepsilon \sim M/L. \tag{2.1.2a}$$

For a source consisting of several lumps, each with mass m and size ℓ, separated by distance $b \gg \ell$,

$\varepsilon \sim m/b$ if one is interested only in waves generated by relative motions of the lumps; (2.1.2b)

$\varepsilon \sim m/\ell$ if one is interested in waves generated by internal motions of the lumps. (2.1.2c)

Weak-field formalisms are valid only for sources that have $\varepsilon \ll 1$.

All weak-field formalisms have roughly the same structure: They utilize a single coordinate system $\{x^\alpha\}$ that covers the entire wave-generation region and local wave zone ($r \lesssim r_0$) and that is as nearly globally Minkowskiian as possible. In this coordinate system they define a "gravitational field" $\bar{h}^{\alpha\beta}$ (second-rank symmetric tensor) by

$$\bar{h}^{\alpha\beta} \equiv - [(-g)^{1/2} g^{\alpha\beta} - \eta^{\alpha\beta}], \qquad (2.1.3)$$

where $g^{\alpha\beta}$ are the contravariant components of the metric tensor, where $g \equiv \det||g_{\alpha\beta}||$, and where $\eta^{\alpha\beta}$ are the components of the Minkowskii metric tensor [diag (-1,1,1,1)]. The formalisms then consist of field equations by which the stress-energy tensor $T^{\alpha\beta}$ (associated with matter and non-gravitational fields) generates the gravitational field $\bar{h}^{\alpha\beta}$, and equations of motion by which the gravitational field and internal stresses regulate the time evolution of the stress-energy tensor.

Thorne and Kovacs (1975; cited henceforth as TK) have devised a classification scheme for weak-field wave-generation formalisms--a scheme resembling that by which Havas and Goldberg (1962) classify "point-mass equation-of-motion formalisms." This scheme characterizes wave-generation formalisms by two integers n_T and n_h. These "order indices" tell one the magnitude of the errors made by the formalism--i.e., the amount by which the formalism's predictions should differ from those of exact general relativity theory:[1]

$$|(\text{errors in } T^{\mu\nu})/T^{00}| \sim \varepsilon^{n_T}, \qquad (2.1.4a)$$

$$|(\text{errors in } \bar{h}^{\mu\nu})/\bar{h}^{00}| \sim \varepsilon^{n_h}. \qquad (2.1.4b)$$

For example, a formalism of order $(n_T, n_h) = (1,1)$ makes fractional errors of order ε in both the stress-energy tensor and the gravitational field, while a formalism of order (2,1) makes fractional errors ε^2 in $T^{\mu\nu}$ and ε in $\bar{h}^{\mu\nu}$.

Errors in $\bar{h}^{\mu\nu}$, when fed into the equation of motion, produce errors in $T^{\mu\nu}$; and similarly, errors in $T^{\mu\nu}$, when fed into the field equations, produce errors in $\bar{h}^{\mu\nu}$. This feeding process places constraints on the order indices (n_T, n_h) of any self-consistent, weak-field wave-generation formalism:

$$n_h = n_T \quad \text{or} \quad n_h = n_T - 1. \qquad (2.1.5)$$

[1]Note that all of the $|T^{\mu\nu}|$ are $\lesssim T^{00}$, and consequently all of the $|\bar{h}^{\mu\nu}|$ are $\lesssim \bar{h}^{00}$. This fact dictates the form of equations (2.1.4).

Thus, every weak-field formalism has order indices of the form (n,n-1) or (n,n) for some integer n.

TK devise a scheme for constructing formalisms of any desired order. They also show that any formalism of order (n,n-1) can readily be "strengthened" by augmenting onto it a higher-accuracy field-generation equation. The resulting, augmented formalism will have order (n,n).

In these lectures I shall not describe the general analysis of TK; I shall merely present examples of weak-field formalisms of various orders, and describe the augmentation process which improves their accuracy.

2.2 Special Relativity: A Formalism of Order (1,0)

Special Relativity is characterized by the equations of motion

$$T^{\mu\nu}{}_{,\nu} = 0, \tag{2.2.1}$$

which are oblivious of all gravitational effects. The largest of the individual terms that occur in these equations are of order

$$T^{00}/\ell, \tag{2.2.2}$$

where ℓ is a characteristic length scale inside the source. By contrast, the gravitational forces ignored by these equations of motion are

$$\Gamma^{\mu}{}_{\alpha\nu}T^{\alpha\nu}+\Gamma^{\nu}{}_{\alpha\nu}T^{\mu\alpha}\sim(\bar{h}^{00}/\ell)T^{00}\sim\varepsilon(T^{00}/\ell). \tag{2.2.3}$$

Comparison of the forces ignored (eq. 2.2.3) with the terms included (eq. 2.2.2) shows that special relativity makes fractional errors of order ε in the evolution of the stress-energy--and hence fractional errors of order ε in the stress-energy tensor itself. Evidently, the stress-energy tensor has order index $n_T = 1$.

The gravitational field, by contrast, has order index $n_h=0$, since special relativity is totally oblivious of gravity and therefore makes fractional errors $\delta\bar{h}^{\mu\nu}/\bar{h}^{00}\sim 1 = \varepsilon^0$.

Conclusion: Special relativity (eq. 2.2.1) is a weak-field formalism of order (1,0).

2.3. Linearized Theory: A Formalism of Order (1,1)

One can obtain Linearized Theory from Special Relativity very easily: One leaves unchanged the stress-energy tensor and its equations of motion

$$T^{\mu\nu}{}_{,\nu} = 0, \tag{2.3.1a}$$

but one postulates that this stress-energy generates a gravitational field $\bar{h}^{\mu\nu}$ by the relation

$$\bar{h}^{\mu\nu}(x) = 16\pi \int G(x,x')T^{\mu\nu}(x')d^4x'. \tag{2.3.1b}$$

Here d^4x' is the special relativistic volume element

$$d^4x' \equiv dx^{0'}dx^{1'}dx^{2'}dx^{3'}, \tag{2.3.2}$$

and $G(x,x')$ is the flat-space propagator (Green's function)

$$G(x,x') = \frac{1}{4\pi}\,\delta_{ret}[1/2(x^\alpha - x^{\alpha'})(x^\beta - x^{\beta'})\eta_{\alpha\beta}], \tag{2.3.3a}$$

$$\delta_{ret} \equiv \begin{cases} \text{Dirac delta function for } x^\alpha \text{ in future of } x^{\alpha'} \\ 0 \qquad\qquad\qquad\qquad\qquad \text{for } x^\alpha \text{ in past of } x^{\alpha'}. \end{cases} \tag{2.3.3b}$$

[Notice: (1) in the arguments, of $\bar{h}^{\mu\nu}$, G, and $T^{\mu\nu}$ we omit the indices of the coordinates x^α and $x^{\alpha'}$; also (2) by integrating over time, $x^{0'}$, in eq. (2.3.1b) one obtains the expression

$$\bar{h}^{\mu\nu}(x^0,x^j) = 4\int \frac{T^{\mu\nu}(x^{0'}=x^0 - |\underset{\sim}{x}-\underset{\sim}{x}'|,\ x^{j'})}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,d^3x', \tag{2.3.4}$$

which is familiar from elementary treatises on Linearized Theory; e.g., chapter 18 of MTW.]

The linearized gravitational field (2.3.1b) is a fairly good approximation to the exact general relativistic gravitational field, $\bar{h}^{\alpha\beta} \equiv -[(-g)^{1/2}g^{\alpha\beta} - \eta^{\alpha\beta}]$. It makes fractional errors of order ε. Hence, Linearized Theory has a gravitational order index $n_h = 1$, which is one order "better" than Special Relativity; but its stress-energy order index is the same as that of Special Relativity, $n_T = 1$. The total order of Linearized Theory is $(n_T, n_h) = (1,1)$.

To within the accuracy of Linearized Theory the metric perturbation $h_{\mu\nu}$ is the trace-reversal of $\bar{h}_{\mu\nu}$:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - 1/2\eta_{\mu\nu}\,\bar{h}; \qquad \bar{h} \equiv \bar{h}_\alpha{}^\alpha. \tag{2.3.5a}$$

(Indices in Linearized Theory are raised and lowered with $\eta_{\mu\nu}$.) Since $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ differ only by a trace, the gravitational-wave field in the local wave zone is (cf. eq. 1.3.3)

$$h^{TT}_{jk} = \bar{h}^{TT}_{jk} = P_{ja}P_{kb}\bar{h}_{ab} - 1/2P_{jk}(P_{ab}\bar{h}_{ab}). \tag{2.3.5b}$$

The fractional errors in this gravitational-wave field are

$$\text{(fractional errors in } \bar{h}^{TT}_{jk}) \equiv \left| \frac{\delta\bar{h}^{TT}_{jk}}{\bar{h}^{TT}_{jk}} \right| = \left| \frac{\delta\bar{h}^{TT}_{jk}}{\bar{h}^{00}} \; \frac{\bar{h}^{00}}{\bar{h}^{TT}_{jk}} \right|$$

$$\sim \frac{\varepsilon}{|\bar{h}^{TT}_{jk}/\bar{h}^{00}|} \sim \frac{\varepsilon}{|\bar{h}^{TT}_{jk}/Mr^{-1}|} \; . \tag{2.3.6}$$

Here $\bar{h}^{00} \sim$ (Newtonian potential) $\sim M/r$ is the largest of the components of the gravitational field; cf. equation (2.1.4b) and footnote 1.

The "rules" for using Linearized Theory to calculate gravitational-wave generation are as follows: (1) Express the special relativistic stress-energy tensor $T^{\mu\nu}$ in terms of the non-gravitational variables of the specific system being analyzed. (2) Solve the special-relativistic equations of motion (2.3.1a) to determine the evolution of the stress-energy tensor. (3) Evaluate the integral (2.3.1b) to determine the first-order gravitational field $\bar{h}^{\mu\nu}$ in the local wave zone. (4) Project out the gravitational-wave field $\bar{h}^{TT}_{jk}$ using equation (2.3.5b). (5) Check, using equation (2.3.6), that the errors in the wave field are acceptably small.

This set of rules shows very clearly the sense in which "Linearized Theory is the theory of order (1,1) obtained by augmenting onto Special Relativity [theory of order (1,0)] field generation equations"; cf. next to last paragraph of § 2.1. In particular, when working in Linearized Theory one at first (Rules 1 and 2) pretends that one is in Special Relativity. Only when one starts evaluating the radiation field (Rules 3 and 4) and its errors (Rule 5) does one depart from Special Relativity.

Recently Press (1977) has transformed the gravitational-wave field (2.3.4) and (2.3.5) of Linearized Theory into the form

$$\bar{h}^{TT}_{jk} = \frac{2}{r}\left\{\frac{d^2}{dt^2}\int [T_{00}+2T_{0p}n_p+T_{pq}n_pn_q]_{ret}x_{j'}\,x_{k'}\,d^3x'\right\}^{TT}. \tag{2.3.7a}$$

Here "ret" means "evaluated at the retarded time"

$$[T_{00}]_{ret} \equiv T_{00}(t-|\underset{\sim}{x}-\underset{\sim}{x}'|,\ \underset{\sim}{x}') \tag{2.3.7b}$$

and n_p is the unit radial vector pointing from the source toward the distant observer,

$$n_p \equiv x^p/r \ . \tag{2.3.7c}$$

Press's expression (2.3.7a) is particularly useful for systems in which $|T_{0j}| \ll T_{00}$ and $|T_{jk}| \ll T_{00}$, since then it involves only the second moment of the retarded energy distribution, $T_{00\,\mathrm{ret}}$. Similar expressions are encountered in the "quadrupole-moment formalism" (eqs. 2.6.4 and 2.6.5 below). However there one requires that the source be confined deep within its own near zone ("slow-motion assumption"), whereas Press's expression requires no such constraint.

Press's expression, appropriately modified, has wider validity than just Linearized Theory: Whenever one has a formalism in which the local-wave-zone field $\bar{h}^{\mu\nu}$ can be written as:

$$\bar{h}^{\mu\nu}(x^0,\underset{\sim}{x})=\frac{4}{r}\int \tau^{\mu\nu}{}_{\mathrm{ret}}\,d^3x', \text{ with } \tau^{\mu\nu}{}_{,\nu}=0. \tag{2.3.8}$$

then (as Press emphasizes) expression (2.3.7a) is valid with $T^{\mu\nu}$ replaced by $\tau^{\mu\nu}$ --whatever that animal may be. The weak-field formalisms classified by TK (§2.1 above) all have this property; cf.§II of TK.

2.3.1. A Valid Application of Linearized Theory

Linearized Theory is applicable whenever one can ignore self-gravitational forces inside the source -- i.e., whenever fractional errors

$$\begin{aligned} |\delta T^{\mu\nu}/T^{00}| &\sim \varepsilon \sim (\text{internal gravitational potential}) \\ |\delta \bar{h}^{TT}_{ij}/\bar{h}^{TT}_{ij}| &\sim \frac{\varepsilon}{|\bar{h}^{TT}_{ij}/\bar{h}^{00}|} \end{aligned} \tag{2.3.9}$$

are acceptable. The following is an example of a valid application of Linearized Theory:

A steel bar of mass $M \sim 500$ tons, length $L \sim 20$ meters, and diameter $D \sim 2$ meters generates gravitational waves by rotating end-over-end with angular velocity $\omega \sim 28$ radians per second [Einstein (1918); Eddington (1922); §36.3 of MTW]. (Faster rotation would tear the bar apart.) For such a bar the internal gravitational field is

$$\varepsilon \sim \frac{M}{D} \sim \left(\frac{5\times10^{8}\,g}{2\times10^{2}\,cm}\right)\times\left(\frac{0.7\times10^{-28}\,cm}{g}\right) \sim 2 \times 10^{-22}. \qquad (2.3.10)$$

The radiation field and the gravitational potential $\bar{h}^{00}$, as calculated from Linearized Theory, turn out to be

$$|\bar{h}^{TT}_{ij}| \sim (ML^2\omega^2/r)\cos[2\omega(t-r) + \text{phase}] \qquad (2.3.11)$$

$$\bar{h}^{00} = 4M/r; \qquad (2.3.12)$$

and the fractional errors in the radiation field (eq. 2.3.6) are thus

$$\left|\frac{\delta\bar{h}^{TT}_{jk}}{\bar{h}^{TT}_{jk}}\right| \sim \frac{\varepsilon}{|\bar{h}^{TT}_{jk}/\bar{h}^{00}|} \sim \frac{\varepsilon}{(\omega L)^2} \qquad (2.3.13)$$

$$\sim \frac{10^{-22}}{[(2\times10^{3}\,cm)(28\,sec^{-1})(3\times10^{10}\,cm/sec)^{-1}]^2} \sim 10^{-11}$$

2.3.2. Invalid Applications of Linearized Theory

For most astrophysical systems internal gravity is important, and Linearized Theory is thus invalid. Examples are as follows:

(1) Gravitational waves from nonradial pulsations of a star. Let the star have mass M and radius R. Then its mean density ρ and its internal gravitational field strength ε are

$$\rho \sim M/R^3, \qquad \varepsilon \sim M/R; \qquad (2.3.14)$$

and its mean pressure [calculated from hydrostatic equilibrium, $dp/dr \sim \rho M/R^2$] is

$$p \sim \rho(M/R) \sim \varepsilon\rho\ . \qquad (2.3.15)$$

Since Linearized Theory makes fractional errors $|\delta T^{\mu\nu}/T^{00}| \sim \varepsilon$, its fractional errors in the pressure are

$$\frac{\delta p}{p} \sim \left|\frac{\delta T^{ij}}{T^{00}}\;\frac{T^{00}}{T^{ij}}\right| \sim \left|\varepsilon\,\frac{\rho}{p}\right| \sim 1. \qquad (2.3.16)$$

Thus, Linearized Theory makes unacceptable errors in the star's internal pressure forces, as well as in its internal gravity.

One should not be surprised to learn that its errors in the gravitational waves due to stellar pulsation are also unacceptably large

$$\left| \frac{\delta \bar{h}^{TT}_{jk}}{\bar{h}^{TT}_{jk}} \right| \gtrsim 1. \tag{2.3.17}$$

(2) Gravitational waves from a near-encounter between two fast-moving stars. Linearized theory predicts no gravitational interaction between the stars. Therefore, according to Linearized Theory, each star proceeds undisturbed along its straight path through flat space, oblivious of the other star. As a result, no gravitational waves are produced -- an obviously incorrect prediction.

2.4 Post-Linear Theory: A Formalism of Order (2,1)

When analyzing the internal structure and motion of systems with significant self gravity, one needs a formalism which makes fractional errors $|\delta T^{\mu\nu}/T^{00}| \lesssim \varepsilon^2$ ---i.e., which has a stress-energy order index $n_T \geq 2$. The least accurate and least complex such formalism is Post-Linear Theory. It has order $(n_T, n_h) = (2,1)$.

For a derivation of Post-Linear Theory from general relativity theory, see Thorne and Kovács (1975; "TK").

In Post-Linear Theory one describes gravity by a gravitational field ${}_1\bar{h}^{\mu\nu}$, which is a good approximation to the general relativistic field (eq. 2.1.3):

$${}_1\bar{h}^{\mu\nu} = - [\sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}][1 + O(\varepsilon)]. \tag{2.4.1}$$

Post-Linear Theory, like Special Relativity and Linearized Theory, utilizes the language of flat spacetime. For example, one raises and lowers indices on the gravitational field by means of the flat Lorentz metric $\eta_{\mu\nu}$; and one performs "trace reversals" in the familiar manner of Linearized Theory:

$${}_1h^{\mu\nu} = {}_1\bar{h}^{\mu\nu} - 1/2\eta^{\mu\nu}\, {}_1\bar{h}\ ; \quad {}_1\bar{h}^{\mu\nu} = {}_1h^{\mu\nu} - 1/2\eta^{\mu\nu}\, {}_1h;$$
$${}_1\bar{h} = -{}_1h = {}_1\bar{h}^{\alpha}_{\alpha} = -\, {}_1h^{\alpha}_{\alpha} \tag{2.4.2}$$

(cf. chapter 18 of MTW). The metric of general relativity is approximated by

$$g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}\ [1 + O(\varepsilon)]. \tag{2.4.3}$$

The stress-energy tensor in Post-Linear Theory will typically be expressed in terms of the electromagnetic field $F_{\mu\nu}$, the matter variables (density, pressure, velocity, viscosity,...), and the gravitational field ${}_1\bar{h}^{\mu\nu}$. It must be constructed from these variables in accordance with all the rules of general relativity, to within fractional errors of $O(\varepsilon^2)$.

The equations of motion of Post-Linear Theory are formally the same as those of general relativity

$$T^{\mu\nu}{}_{,\nu} + {}_1\Gamma^{\mu}{}_{\alpha\nu}\, T^{\alpha\nu} + {}_1\Gamma^{\nu}{}_{\alpha\nu}\, T^{\mu\alpha} = 0; \tag{2.4.4a}$$

however, the Christoffel symbols ${}_1\Gamma^{\mu}{}_{\alpha\beta}$ are given by the "first-order approximation"

$${}_1\Gamma^{\mu}{}_{\alpha\beta} = \eta^{\mu\nu}\, {}_1\Gamma_{\nu\alpha\beta}\ ;\ {}_1\Gamma_{\nu\alpha\beta} = ({}_1h_{\nu\alpha,\beta} + {}_1h_{\nu\beta,\alpha} - {}_1h_{\alpha\beta,\nu}). \tag{2.4.4b}$$

The field equations of Post-Linear Theory are the same as those of Linearized Theory

$${}_1\bar{h}^{\mu\nu,\alpha}{}_{\alpha} = -\,16\pi\, T^{\mu\nu}; \tag{2.4.4c}$$

and their solution can be written, using the flat-space propagator (eq. 2.3.3), as

$${}_1\bar{h}^{\mu\nu}(x) = 16\pi \int G(x,x')\,T^{\mu\nu}(x')\,d^4x'. \tag{2.4.5}$$

Notice that Post-Linear Theory differs from Linearized Theory in one crucial, but simple way: It allows the gravitational field ${}_1\bar{h}^{\mu\nu}$ to "push the matter around". [Christoffel symbols have been inserted into the equation of motion; compare eqs. (2.3.1a) and (2.4.4a).] This difference is crucial for astrophysics. It allows Post-Linear Theory to treat accurately the structure and evolution of stars, planets, planetary systems, star clusters, and near stellar encounters -- unless the stars are highly compact (i.e., unless they are neutron stars or black holes). Linearized Theory makes enormous errors on all such systems.

2.5. Post-Linear Wave Generation: A Formalism of Order (2,2)

Despite its fine ability to analyze the internal dynamics of astrophysical systems, Post-Linear Theory does a bad job of predicting their gravitational-wave generation. For wave generation it has no better accuracy than Linearized Theory.

Fortunately, there is a simple way to improve its accuracy. One need only append to it a gravitational field ${}_2\bar{h}^{\mu\nu}$, which is more accurate than ${}_1\bar{h}^{\mu\nu}$:

$$ {}_2\bar{h}^{\mu\nu} = -[\sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}][1 + 0\ (\varepsilon^2)]. \qquad (2.5.1)$$

By appending ${}_2\bar{h}^{\mu\nu}$ onto Post-Linear Theory, one boosts its accuracy from order (2,1) to order (2,2). The resulting formalism, called the "Post-Linear Wave-Generation Formalism," bears the same relationship to Post-Linear Theory as Linearized Theory does to Special Relativity.

TK have derived a formula for ${}_2\bar{h}^{\mu\nu}$ in terms of ${}_1\bar{h}^{\mu\nu}$ and $T^{\mu\nu}$. Their formula involves the flat-space propagator

$$G(x,x') = \frac{1}{4\pi}\ \delta_{ret}\ [1/2(x^{\alpha}-x^{\alpha'})(x^{\beta}-x^{\beta'})\eta_{\alpha\beta}] \qquad (2.5.2a)$$

[eq. (2.3.3)], and also its derivative with respect to the argument of the delta function

$$G'(x,x') = \frac{1}{4\pi}\ \delta'_{ret}[1/2\ (x^{\alpha}-x^{\alpha'})(x^{\beta}-x^{\beta'})\eta_{\alpha\beta}]; \qquad (2.5.2b)$$

$$\delta'_{ret}(z) \equiv (d/dz)\ \delta_{ret}(z).$$

The TK formula splits ${}_2\bar{h}^{\mu\nu}$ into five pieces

$$ {}_2\bar{h}^{\mu\nu} = {}_2\bar{h}^{\mu\nu}_D + {}_2\bar{h}^{\mu\nu}_F + {}_2\bar{h}^{\mu\nu}_{TL} + {}_2\bar{h}^{\mu\nu}_{TR} + {}_2\bar{h}^{\mu\nu}_W . \qquad (2.5.3)$$

Each piece has a particular physical interpretation. However, one must be careful not to take that interpretation too seriously -- for reasons discussed below. The five pieces are as follows:

Direct field, ${}_2\bar{h}^{\mu\nu}_D$. This field is produced directly by the Post-Linear stress-energy tensor $T^{\mu\nu}$; and it propagates from the source point $x^{\alpha'}$ to the field point x^{α} by means of the flat-space propagator. In other words, it propagates along the flat-space light cone with parallel-propagation of components and with a "1/r" fall-off of amplitude. The formula for this direct field is

$$ {}_2\bar{h}^{\mu\nu}_D\ (x) = 16\pi \int G(x,x')T^{\mu\nu}(x')[1-{}_1\bar{h}(x')]d^4x'. \qquad (2.5.4)$$

This part of ${}_2\bar{h}^{\mu\nu}$ is $\sim M/r$, whereas the other four parts are typically $\lesssim M/r$.

Focussing field, ${}_2\bar{h}^{\mu\nu}_F$. When the gravitational field generated at $x^{\alpha'}$ propagates through regions of nonzero Ricci curvature -- i.e., through matter --, it gets focussed. This focussing increases the amplitude of ${}_2\bar{h}^{\mu\nu}$, without changing its directionality (i.e., without changing the relative magnitude of its components). The amount of focussing between a source point x^{α} and a field point x^{α} is described by the "focussing function"

$$\alpha(x,x') \equiv 1/2\; X^{\alpha}X^{\beta}\int_0^1 {}_1R_{\alpha\beta}(x^{\mu'} + \lambda X^{\mu})\lambda(1-\lambda)d\lambda, \qquad (2.5.5a)$$

$$X^{\alpha} \equiv x^{\alpha} - x^{\alpha'}.$$

Here ${}_1R_{\alpha\beta}(x^{\mu'} + \lambda X^{\mu})$ is the first-order Ricci tensor, (calculated from $g_{\alpha\beta} = \eta_{\alpha\beta} + {}_1h_{\alpha\beta}$), evaluated at the event $x^{\mu'} + \lambda X^{\mu}$, which lies a fraction λ of the way along the straight line between source point and field point. A formula for ${}_1R_{\alpha\beta}$ is

$${}_1R_{\alpha\beta} = -1/2\; {}_1h_{\alpha\beta,\rho}{}^{\rho} = 8\pi(T_{\alpha\beta} - 1/2\eta_{\alpha\beta}T). \qquad (2.5.5b)$$

For intuition into the focussing function, see Figure 3. In terms of the focussing function α, the flat-space propagator G, and the stress-energy tensor $T^{\mu\nu}$, the focussing field is

$${}_2\bar{h}_F^{\mu\nu}(x) = 16\pi\int \alpha(x,x')G(x,x')T^{\mu\nu}(x')d^4x'. \qquad (2.5.5c)$$

<u>Tail Field, ${}_2\bar{h}_{TL}^{\mu\nu}$</u>. Consider the gravitational field ${}_2\bar{h}^{\mu\nu}$ generated at an event $x^{\alpha'}$. It has a "wave front" that propagates outward, initially spherically and initially along the future light cone of $x^{\alpha'}$. However, focussing produces dimples in the wave front; and dimpling, when analyzed from the viewpoint of Huygens' principle, produces waves that radiate outward from the dimpled region in all directions. (See Figure 4.) The result is a "tail" of the wave field. Let a wave originating at $x^{\alpha'}$ arrive at an event $x^{\alpha''}$, with a dimple in its wave front due to focussing. The amplitude $\beta(x'', x')$ for this dimple to produce a tail is given by

$$\beta(x'',x') = X^{\alpha''}X^{\beta''}\int_0^1 {}_1R_{\alpha\beta}(x^{\mu'} + \lambda X^{\mu''})\lambda^2 d\lambda, \qquad (2.5.6a)$$

$$X^{\alpha''} \equiv x^{\alpha''} - x^{\alpha'}. \qquad (2.5.6b)$$

This amplitude is called the "tail-generating function." The tail which it generates is given by

$$\bar{h}_{TL}^{\mu\nu}(x) = -16\pi \iint_{x'\in I^-(x)} G(x,x'')\beta(x'',x')G'(x'',x')T^{\mu\nu}(x')d^4x''d^4x'. \qquad (2.5.6c)$$

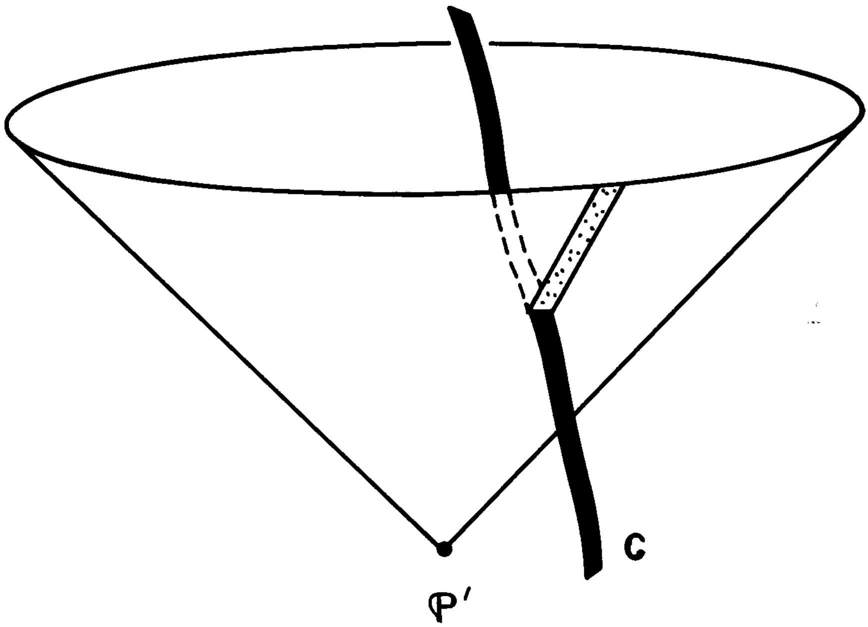

Figure 3. The region of focussing for a gravitational field generated at $\mathcal{P}'$, which propagates through spacetime containing a lump of matter (world tube $\mathcal{C}$). Focussing occurs , and α and $\bar{h}_F^{\mu\nu}$ are nonzero, in the stippled region--i.e., in the region containing rays that have passed through the lump.

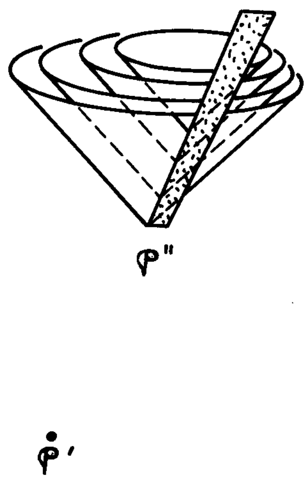

Figure 4. The tail of the gravitational field described in Figure 3. In the region of focussing (stippled region) the tail-generating function β is non-zero. Thus, each point $\mathcal{P}''$ in the stippled region is a source of tail -- and the tail propagates outward from each such $\mathcal{P}''$ along its future light cone.

Here the expression $x' \epsilon I^-(x)$ means that the integration must extend over source points $x^{\alpha'}$ that lie inside but not on the past (flat-space) light cone of x^α.

Transition Field, ${}_2\bar{h}^{\mu\nu}_{TR}$. The gravitational field ${}_2\bar{h}^{\mu\nu}$ generated at an event $x^{\alpha'}$ does not really propagate along light cones of the (fictitious) flat space of Post-Linear Theory. Rather, it propagates along light cones of the slightly curved metric $g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}$. The result is a time delay relative to flat-space propagation -- a delay measured, for example, in the "Shapiro radar time-delay experiment" (Shapiro 1964;§40.4 of MTW). The direct field (eq. 2.5.4) fails to take this time delay into account. Therefore, one must correct it for the effects of the time delay. The "transition field" ${}_2\bar{h}^{\mu\nu}_{TR}$ does that correcting. The correction is embodied in a "time-delay function"

$$\gamma(x,x') = 1/2X^\alpha X^\beta \int_0^1 {}_1h_{\alpha\beta}(x^{\mu'} + \lambda X^\mu)d\lambda, \qquad (2.5.7a)$$

$$X^\mu \equiv x^\mu - x^{\mu'}. \qquad (2.5.7b)$$

For two events x and x' separated by a distance ℓ in the laboratory frame, this time-delay function is

$$\gamma(x,x') = \ell \Delta t_s, \qquad (2.5.7c)$$

where Δt_s is the "Shapiro time delay" (difference between curved-space and flat-space propagation times). In terms of this time-delay function, the transition field is given by[2]

$$\bar{h}^{\mu\nu}_{TR}(x) = 16\pi \int \gamma(x,x')G'(x,x')T^{\mu\nu}(x')d^4x'. \qquad (2.5.7d)$$

The name"transition field" is taken from electromagnetic theory: When a charged particle moves, with uniform velocity, through a medium of variable index of refraction, it radiates. The radiation (called "electromagnetic transition radiation") is caused by variations in the speed of propagation of the particle's Coulomb field.

[2]The expression for $\bar{h}^{\mu\nu}_{TR}$ given by TK differs slightly from eq.(2.5.7d): TK have to "truncate the external time delay" because they do not restrict attention to the local wave zone, wher the external delay is negligible.

The "gravitational transition field" (2.5.7d) has a similar origin: It is caused by variations in the "flat-space" speed of propagation of the particle's direct ("Coulomb") gravitational field.

Whump field, ${}_2\bar{h}^{\mu\nu}_W$. The gravitational field ${}_2\bar{h}^{\mu\nu}$ is generated not only by the material stress-energy tensor $T^{\mu\nu}$, but also by "gravitational stresses." For example, when two stars go flying past each other at high speed, their relative gravitational potential energy builds up quickly, then dies out quickly (i.e., goes "whump"), and in doing so it produces a burst of second-order gravitational field. TK call this field the "whump field", not only in the case of stellar fly-by, but also in general. The "gravitational stresses" $W^{\mu\nu}$ which generate it are a certain sum of products of first derivatives of ${}_1\bar{h}^{\alpha\beta}$; in particular,

$$W^{\mu\nu} \equiv {}_1t^{\mu\nu}_{LL} + (16\pi)^{-1}\ {}_1\bar{h}^{\mu\rho}{}_{,\sigma}\ {}_1\bar{h}^{\nu\sigma}{}_{,\rho} \tag{2.5.8a}$$

where ${}_1t^{\mu\nu}$ is the "Landau-Lifshitz pseudotensor," accurate up to fractional errors $\sim \varepsilon$:

$$\begin{aligned} {}_1t^{\alpha\beta}_{LL} = (16\pi)^{-1}\Big\{&\tfrac{1}{2}\eta^{\alpha\beta}\ {}_1\bar{h}_{\lambda\nu,\rho}\ {}_1\bar{h}^{\lambda\rho,\nu} + {}_1\bar{h}^{\alpha}{}_{\lambda,\nu}\,{}_1\bar{h}^{\beta\lambda,\nu} \\ &-({}_1\bar{h}_{\nu\rho}{}^{,\alpha}\ {}_1\bar{h}^{\beta\nu,\rho} + {}_1\bar{h}_{\nu\rho}{}^{,\beta}\ {}_1\bar{h}^{\alpha\nu,\rho}) + \tfrac{1}{2}\,{}_1\bar{h}_{\lambda\nu}{}^{,\alpha}\ {}_1\bar{h}^{\lambda\nu,\beta} \\ &-\tfrac{1}{4}\eta^{\alpha\beta}\ {}_1\bar{h}_{\lambda\nu,\rho}\ {}_1\bar{h}^{\lambda\nu,\rho} - \tfrac{1}{4}\,{}_1\bar{h}^{,\alpha}\ {}_1\bar{h}^{,\beta} + \tfrac{1}{8}\eta^{\alpha\beta}\ {}_1\bar{h}_{,\lambda}\,{}_1\bar{h}^{,\lambda}\Big\}. \end{aligned} \tag{2.5.8b}$$

In terms of these stresses, the whump field is

$${}_2\bar{h}^{\mu\nu}_W(x) = 16\pi\int G(x,x')W^{\mu\nu}(x')d^4x'. \tag{2.5.8c}$$

The gravitational waves emitted by a Post-Linear source are described by the "tranverse, traceless part" of ${}_2\bar{h}^{\mu\nu}$, evaluated in the local wave zone

$$h^{TT}_{jk} = \bar{h}^{TT}_{jk} = P_{ja}\ {}_2\bar{h}_{ab}\ P_{bk} - \tfrac{1}{2}P_{jk}(P_{ab}\ \ {}_2\bar{h}_{ab}). \tag{2.5.9}$$

Here P_{jk} is the usual transverse projection tensor(eq. 1.3.4).

In the local wave zone ${}_2\bar{h}^{\mu\nu}$ satisfies the flat-space Einstein field equations ${}_2\bar{h}^{\mu\nu}{}_{,\alpha\beta}\eta^{\alpha\beta} = 0$ and the flat-space "Lorentz gauge condition" ${}_2\bar{h}^{\mu\nu}{}_{,\nu} = 0$, except for fractional errors of $O(\varepsilon^2)$.

As a consequence, the gravitational-wave field $\bar{h}^{TT}_{jk}$

transforms, under Lorentz transformations, in the usual manner for a field of spin two and zero rest mass.

Unfortunately, each of the five individual pieces of ${}_2\bar{h}^{\mu\nu}$, by itself, fails to satisfy the gauge condition. As a consequence, the transverse-traceless part of each piece (e.g., $[{}_2\bar{h}^{jk}_{TL}]^{TT}$) fails to transform as a spin-two, zero-rest-mass field. This fact prevents one from attributing a rigorous physical significance to each of the five pieces by itself. The physical interpretations given above are therefore somewhat heuristic. For further discussion, see TK, and also §VIc of Kovács and Thorne (1977a).

Crowley and Thorne (1977; their eq. 37) have derived several potentially useful formulas for the sum of the focussing, tail, and transition fields. An example is:

$$
{}_2\bar{h}^{\mu\nu}_{F}(x) + {}_2\bar{h}^{\mu\nu}_{TL}(x) + {}_2\bar{h}^{\mu\nu}_{TR}(x) = = -\frac{\partial}{\partial x^{\alpha}} \int G(x,x')\, {}_1\bar{h}^{\alpha\beta}(x') \frac{\partial}{\partial x^{\beta}}\, {}_1\bar{h}^{\mu\nu}(x')\, d^4x' . \qquad (2.5.10)
$$

Crowley and Thorne also examine the issue of when one can combine the Post-Linear Wave-Generation Formalism with a point-particle description of gravitating bodies. The answer is non-trivial: Some sets of post-linear formulas are compatible with point-particle descriptions of matter; others are not.

The accuracy of the Post-Linear Wave-Generation Formalism is a slightly delicate issue. Until one looks closely, one expects

$$
(\text{fractional errors in } {}_2\bar{h}^{\mu\nu}) \sim \varepsilon^2 ; \qquad (2.5.11a)
$$

$$
(\text{fractional errors in } {}_2\bar{h}^{TT}_{jk}) \sim \frac{\varepsilon^2}{|\bar{h}^{TT}_{jk}/\bar{h}^{00}|} . \qquad (2.5.11b)
$$

However, the formalism as expounded above relies on two approximations which can sometimes produce larger errors than (2.5.11): First, it assumes that focussing is small -- i.e., that the focussing function satisfies

[3]See, e.g., §III.B of Eardley, Lee, and Lightman (1974), with the change of notation $\mathbf{e}_{\hat{z}} \to \mathbf{n} = \mathbf{e}_r$ = (unit radial vector), $\mathbf{e}_{\hat{x}} \to \mathbf{e}_{\hat{\theta}}$ = (unit vector in θ direction), $\mathbf{e}_{\hat{y}} \to \mathbf{e}_{\hat{\varphi}}$ = (unit vector in φ direction), and with specialization to the spin-2 case, $\Psi_2 = \Psi_3 = \Phi_{22} = 0$; and also Thorne (1977d).

$$|\alpha(x,x')| \ll 1 \text{ for all } x' \text{ inside source and } x \text{ in local wave zone.} \qquad (2.5.12)$$

Second, it assumes that the Shapiro time delays inside the source are short compared to the characteristic timescale for the source's motion -- i.e., that Δt_s (eq. 2.5.7c) and the reduced wavelength of the radiation, $\lambdabar$, satisfy

$$\Delta t_s \ll \lambdabar \, . \qquad (2.5.13)$$

These assumptions could be dropped, but only at the cost of making the formalism somewhat more complicated than it already is.

For realistic sources the "short-Shapiro-time-delay" assumption (2.5.13) is generally satisfied. However, one can readily imagine interesting weak-field sources ($\varepsilon \ll 1$) which violate the "small-focussing assumption." For example, for two stars of mass m and size ℓ separated by a distance $b \gg \ell$, the focussing function for rays originating in one star and passing through the other is

$$\alpha \sim (b/\ell)(m/\ell) \sim 10^{-6}\,(b/\ell)(m/M_\odot)(\ell/R_\odot)^{-1} \, . \qquad (2.5.14)$$

This can be $\gtrsim 1$ for sufficiently large separations b.

The small-focussing and short-Shapiro-time-delay assumption force one to modify the error estimates (2.5.11). The correct error estimates are

$$(\text{fractional errors in } {}_2\bar{h}^{\mu\nu}) \sim \text{maximum}\,(\varepsilon^2, \varepsilon\alpha, \varepsilon\Delta t_s/\lambdabar), \qquad (2.5.15a)$$

$$(\text{fractional errors in } {}_2\bar{h}^{TT}_{jk}) \sim \frac{\text{maximum}\,(\varepsilon^2, \varepsilon\alpha, \varepsilon\Delta t_s/\lambdabar)}{|\bar{h}^{TT}_{jk}/\bar{h}^{00}|} \, . \qquad (2.5.15b)$$

The "rules" for using the Post-Linear Wave-Generation Formalism are as follows: (1) Express the Post-Linear stress-energy tensor $T^{\mu\nu}$ in terms of the nongravitational variables of the specific system being analyzed, and in terms of the first-order gravitational field ${}_1\bar{h}^{\mu\nu}$. Do so in the manner of general relativity, up to fractional errors of $O(\varepsilon^2)$. (2) Solve the coupled Post-Linear equations of motion (2.4.4a) and Post-Linear gravitational field equations (2.4.4c) for the evolution of the source and its first-order gravitational field ${}_1\bar{h}^{\mu\nu}$. (3) Evaluate the integrals (2.5.3)-(2.5.8) for field points in the local wave zone to determine th five pieces of the second-order gravitational field ${}_2\bar{h}^{\mu\nu}$. Project out the gravitational-wave field $\bar{h}^{TT}_{jk}$ in the local wave zone using equation (1.3.3) (5) Check, using equation (2.5.15b), that the errors in the wave field are acceptably small.

2.5.1. Sample Applications

The Post-Linear Wave-Generation Formalism can be applied, with good accuracy, to most astrophysical systems. For example, it can be applied to the two systems of §2.3.2, in which Linearized Theory was a failure: pulsations of a noncompact star, and near-encounters between fast-moving stars.

Actually, for the stellar pulsation problem -- and for any other problem characterized by slow internal motions (speeds much less than that of light) -- one need not resort to the Post-Linear Formalism in its full complexity. Rather, one can apply a slow-motion variant of the formalism (§2.6 below).

However, for systems with fast motion and weak but significant internal gravity, the Post-Linear Formalism is the simplest formalism that will do the full job. An example is "gravitational Bremsstrahlung radiation" produced in a near-encounter between nearly Newtonian stars ($M/R \ll 1$), which fly past each other with a relative velocity $v \gtrsim 0.1$ x (speed of light). This example is treated in detail by Kovács and Thorne (1977a,b). [Limiting cases and special features of the Bremsstrahlung problem are actually amenable to other techniques, which I do not review in these lectures. These include the Feynman-diagram method (originally designed for quantum gravity problems, but also applicable to classical problems; see Feynman 1963); the method of virtual quanta (Matzner and Nutku 1974); the method of Green's functions for the linearized Schwarzschild metric (a special case of the perturbation methods of §5 below; Peters 1970); and a new colliding-plane-waves tecnique (D'Eath 1977)].

2.6. The Slow-Motion Limit of the Post-Linear Formalism: Newtonian Theory [Order (2,1)] and Quadrupole-Moment Formalism [Order (2,2)]

Most astrophysical systems have not only weak internal gravity, $\varepsilon \ll 1$, but also slow internal motions and weak internal stresses. For such systems the Post-Linear Formalism of §2.4, which has order (2,1), reduces to the Newtonian Theory of Gravity; and the Post-Linear Wave-Generation Formalism of §2.5, which has order (2,2), reduces to the Quadrupole-Moment Wave-Generation Formalism. I shall briefly summarize these formalisms and their realms of validity. For details of the slow-motion transition from the Post-Linear Formalism to these formalisms, see TK. For derivations of these formalisms assuming slow motion from the beginning see, e.g., § 104 of Landau and Lifshitz (1962) or chapter 36 of MTW.

Realm of validity: One can characterize a slow-motion, weak-field system by the following parameters:

$$
\begin{aligned}
M &\equiv \text{(mass of system)} \\
L &\equiv \text{(size of system)} \\
\lambda\!\!\!^{-} &\equiv \text{(characteristic timescale of system)} \\
&= \text{(reduced wavelength of radiation)} \\
\mathbf{v} &\equiv (|T^{0j}|/T^{00})_{max} = \text{(maximum internal velocity)} \qquad (2.6.1)\\
S &\equiv (|T^{ij}|/T^{00})_{max} = \text{(maximum of stress/density)} \\
\varepsilon &\equiv \text{(magnitude of internal gravity)} \sim M/L.
\end{aligned}
$$

For a static system $\lambda\!\!\!^{-}$ is not zero; rather, it is the characteristic timescale for perturbations against which one hopes the system is stable (e.g., for small-amplitude pulsations, if the system is a static star).

Newtonian Gravitation Theory and the Quadrupole-Moment Formalism require for their validity the following conditions:

Weak gravity:	$\varepsilon \ll 1$	(2.6.2a)
Slow motion:	$L/\lambda\!\!\!^{-} \ll 1$, which implies $v \ll 1$	(2.6.2b)
Small Stresses:	$S^2 \ll 1$,	(2.6.2c)
System not near marginal stability:	$\begin{cases} \omega^2 \equiv (1/\lambda\!\!\!^{-})^2 \gg (M/L)(M/L^3) \\ \omega^2 \equiv (1/\lambda\!\!\!^{-})^2 \gg S^2 (S/L)^2 \end{cases}$	(2.6.2d)

[If the last condition is violated, then post-Newtonian gravitational forces may affect significantly the stability of the system and its motion; cf. Chandrasekhar (1964), or Boxes 24.2 and 26.2 of MTW.]

Newtonian Theory Summarized: Newtonian Theory describes gravity by a scalar gravitational potential U, and describes matter by its mass density ρ, its velocity v_j, and the stress t_{jk} measured in its rest frame. The equations of the theory are: (1) conservation of mass

$$\rho_{,t} + (\rho v_j)_{,j} = 0; \qquad (2.6.3a)$$

(2) "Euler" equations of motion

$$\rho(v_{j,t} + v_{j,k} v_k) = \rho U_{,j} - t_{jk,k}; \qquad (2.6.3b)$$

(3) gravitational field equation

$$U_{,jj} = -4\pi\rho, \qquad (2.6.3c)$$

which has the solution

$$U(\underset{\sim}{x},t) = \int \frac{\rho(\underset{\sim}{x}',t)}{\left|\underset{\sim}{x} - \underset{\sim}{x}'\right|} d^3x'. \qquad (2.6.3d)$$

For further details see, e.g., §39.7 of MTW.

Quadrupole-Moment Formalism Summarized: To calculate the gravitational waves from a nearly Newtonian system, one can proceed as follows: (1) Specify, by an equation of state, or viscosity, or some other method, the dependence of the stresses t_{jk} on ρ, v_j, and other variables of the system. (2) Solve the Newtonian equations (2.6.3) to determine the structure and evolution of the system. (3) Calculate the gravitational-wave amplitude $\bar{h}^{TT}_{jk}$ in the local wave zone using the following formula:

$$h^{TT}_{jk} = \frac{2}{r}\left[\ddot{\mathcal{I}}_{jk}(t-r)\right]^{TT}. \qquad (2.6.4)$$

Here $\mathcal{I}_{jk}(t)$ is the "reduced quadrupole moment" of the source at time t

$$\mathcal{I}_{jk}(t) = \int \rho(t,\underset{\sim}{x}')\left[x^{j'} x^{k'} - \frac{1}{3}\delta^{jk}(\underset{\sim}{x}')^2\right]d^3x'; \qquad (2.6.5)$$

$\ddot{\mathcal{I}}_{jk}$ is the second time derivative of $\mathcal{I}_{jk}$; r is distance from the source to the field point; and "TT" denotes the transverse-traceless projection process of equation (1.3.3). (4) Verify that the "conditions of validity" (2.6.2) are satisfied.

2.6.1. Sample Applications

The quadrupole-moment formalism has been used more widely in gravitational-wave calculations than any other wave-generation formalism. Examples of its application are: the waves emitted by quadrupole pulsations of a star [see, e.g., Wheeler (1966)]; the waves emitted by orbital motion of a binary star system [see, e.g., Peters and Mathews (1963)]; the gravitational bremsstrahlung radiation from low-velocity near-encounters of two stars [low velocity limit of the problem described in §2.5.1; see, e.g., Ruffini and Wheeler (1971)]; and the "tidal gravitational radiation" produced when a Newtonian star is tidally deformed by passage near a black hole or other star [Mashoon (1973), Turner (1977)].

2.7 The Post-Newtonian Wave-Generation Formalism [Order (3,3)]

For systems with slow, but not extremely slow motions (e.g., $v \sim 0.3$), and weak, but not extremely weak gravity (e.g., $\varepsilon \sim 0.1$), one needs a gravitational-wave formalism of higher accuracy than the Quadrupole-Moment Formalism. More specifically, one needs a formalism that takes account of Post-Newtonian effects in the structure and evolution of the system.

Epstein and Wagoner (1975) have devised such a formalism. Like all Post-Newtonian formalisms, it is valid only for gravitationally bound, or nearly gravitationally bound systems -- i.e., for systems with

$$v^2 \lesssim \varepsilon, \qquad S^2 \lesssim \varepsilon \tag{2.7.1a}$$

[cf. eq. (2.6.1)]; and it requires moderately weak internal gravity

$$\varepsilon \lesssim 0.1. \tag{2.7.1b}$$

The Epstein-Wagoner "Post-Newtonian Wave-Generation Formalism" has the following structure: (1) First one calculates the structure and evolution of the source using a Post-Newtonian Formalism of order $(n_T, n_h) = (3,2)$ -- e.g., the perfect-fluid Post-Newtonian formalism of Chandrasekhar (1965). (2) Then one evaluates the radiation field in the local wave zone using formulas to be given below (§4.5). [These formulas boost the order of the formalism up to $(n_T, n_h) = (3,3)$.] (3) Finally one verifies that the emitting system satisfies the Post-Newtonian validity conditions (2.7.1).

I delay presenting the Epstein-Wagoner equations until after I have treated multipole-moment formalisms, because Epstein and Wagoner utilize multipole moments in a nontrivial fashion.

2.7.1. Sample Applications

The Epstein-Wagoner formalism is being used in two different ways by the Stanford relativity group: (i) They use it to study deviations from the quadrupole-moment formalism for systems with moderately strong internal gravity or moderately high velocities or both [e.g. for moderately relativistic binary systems, and for stellar flybies at moderately high velocity (gravitational bremsstrahling); see Wagoner and Will (1976)]. (ii) They use it to seek insight into highly relativistic situations [e.g. waves from the collapse of a rotating star to form a black hole; Epstein (1976), Wagoner (1977)], where one can hope for, but is not guaranteed, more reliable results than from the quadrupole-moment formalism.

3. MULTIPOLE-MOMENT ANALYSIS OF THE RADIATION FIELD

3.1. Gravitational-Wave Field

I now wish to turn attention from weak-field formalisms to slow-motion formalisms. But before doing so, I must digress into a topic which is needed as an underpinning for the slow-motion discussion. This topic is the multipole structure of the gravitational-wave field for any isolated source. A number of researchers (not including me) have made major contributions to this subject; see end of §3.3 for references. Recently, I have taken the various contributions, have exhibited the relationships between some of their notations, and have tried to consolidate them into a single formalism in which (to me) the notation looks optimal. The following review of the subject is based on that work (Thorne 1977a--which I shall cite henceforth as TV, since it is Paper V in a series).

Consider the gravitational-wave field h_{jk}^{TT} in the local wave zone of an isolated source. The local wave zone is so constructed [eqs. (1.2.2)-(1.2.5)] that in it one can regard the background on which the waves propagate as completely flat. Consequently, the waves have the radially-propagating, flat-space form

$$h_{jk}^{TT} = r^{-1} A_{jk}(t - r, \theta, \phi) \tag{3.1.1}$$

where A_{jk} is a symmetric, transverse, traceless amplitude that depends on retarded time $u = t - r$ and on angular location (θ, ϕ) around the source. The fact that A_{jk}^{TT} is symmetric, transverse, and traceless is embodied in the algebraic relations

Symmetry: $A_{jk} = A_{kj}$, (3.1.2a)

Transverse: $A_{jk} n_k = 0$, where $\underset{\sim}{n} \equiv \underset{\sim}{x}/r$ = unit radial vector, (3.1.2b)

Traceless: $A_{jj} = 0$. (3.1.2c)

It is often useful to expand the wave amplitude A_{jk} in spherical harmonics, and to identify the coefficients of the expansion as the multipole moments of the radiation field. In these lectures I shall follow the notation of Sachs (1961), Pirani (1965), MTW (Chap. 36), and TV (Thorne 1977a), which uses symmetric, trace-free tensors to represent spherical harmonics. The formalism of symmetric, trace-free tensors is described briefly in the next section.

3.2. Symmetric, Trace-Free Tensors and Spherical Harmonics

Let $\underline{A}$ be a symmetric, constant (position-independent) tensor of rank ℓ defined in 3-dimensional, flat space. We shall denote its components in a Cartesian coordinate system by $A_{k_1 \ldots k_\ell}$, and the components of its completely symmetric part by $A_{(k_1 \ldots k_\ell)}$:

$$A_{(k_1 \ldots k_\ell)} \equiv \frac{1}{\ell!} \sum_{\pi} A_{k_{\pi(1)} \ldots k_{\pi(\ell)}} . \qquad (3.2.1)$$

Here the summation goes over all $\ell!$ permutations, π, of $1 \ldots \ell$. We shall denote the trace-free, symmetric part of $\underline{A}$ by the corresponding capital script letter, $\mathcal{A}$:

$$\mathcal{A}_{k_1 \ldots k_\ell} = \sum_{n=0}^{[\ell/2]} (-1)^n \frac{\ell!(2\ell-2n-1)!!}{(\ell-2n)!(2\ell-n)!!(2n)!!} \times$$
$$\times \delta_{(k_1 k_2} \ldots \delta_{k_{2n-1} k_{2n}} S_{k_{2n+1} \ldots k_\ell) j_1 j_1 \ldots j_n j_n} . \qquad (3.2.2)$$

Here δ_{ab} = (Kronecker delta) are the components of the metric, $[\ell/2]$ means the largest integer less than or equal to $\ell/2$, $S_{k_1 \ldots k_\ell} \equiv A_{(k_1 \ldots k_\ell)}$, and the "double factorial" (2n)!! is defined by (3.2.8) below.

Consider the set of all symmetric, trace-free tensors of rank ℓ ("STF-ℓ tensors"). The STF-ℓ tensors generate an irreducible representation of the rotation group, of weight ℓ. Hence, there exists a one-to-one mapping between them and the spherical harmonics of order ℓ. To exhibit that mapping we introduce the unit radial vector $\underset{\sim}{n}$ of our Euclidean space, and we express its Cartesian components in terms of spherical coordinates, θ and ϕ :

$$\underset{\sim}{n}=\underset{\sim}{x}/|\underset{\sim}{x}|, \quad n_1+in_2=\sin\theta\ e^{i\phi}, \quad n_3=\cos\theta. \tag{3.2.3}$$

From the usual expression for the spherical harmonic $Y^{\ell m}(\theta,\phi)$ in terms of $\cos\theta$ and $\sin\theta\ e^{i\phi}$, it is easy to show [see pp. 289-290 of Pirani (1965) or see TV] that

$$Y^{\ell m}(\theta,\phi)=\mathcal{Y}^{\ell m}_{k_1\ldots k_\ell}\, n_{k_1}\ldots n_{k_\ell}. \tag{3.2.4}$$

Here $\mathcal{Y}^{\ell m}_{k_1\ldots k_\ell}$ is the following STF-ℓ tensor

$$\mathcal{Y}^{\ell m}_{k_1\ldots k_\ell} \equiv C^{\ell m} \sum_{j=0}^{[(\ell-m)/2]} a^{\ell m j} (\delta^1_{k_1}+i\delta^2_{k_1})\ldots(\delta^1_{k_m}+i\delta^2_{k_m})\,.$$

$$.\,\delta^3_{k_{m+1}}\ldots\delta^3_{k_{\ell-2j}}(\delta^{a_1}_{k_{\ell-2j+1}}\delta^{a_1}_{k_{\ell-2j+2}})\ldots(\delta^{a_j}_{k_{\ell-1}}\delta^{a_j}_{k_\ell}) \text{ for } m\geq 0, \tag{3.2.5}$$

$$\mathcal{Y}^{\ell m}_{k_1\ldots k_\ell} \equiv (-1)^m (\mathcal{Y}^{\ell\,-m}_{k_1\ldots k_\ell})^* \quad \text{for} \quad m<0.$$

$$C^{\ell m} \equiv (-1)^m \left[\frac{2\ell+1}{4\pi}\ \frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}, \quad a^{\ell m j} \equiv \frac{(-1)^j(2\ell-2j)!}{2^\ell j!(\ell-j)!(\ell-m-2j)!}.$$

Here $[(\ell-m)/2]$ means the largest integer less than or equal to $(\ell-m)/2$, and $*$ means complex conjugate.

Henceforth, to simplify notation, when we encounter a sequence of many (say ℓ) indices on a tensor we shall denote it as follows

$$S_{A_\ell} \equiv S_{a_1\ldots a_\ell}. \tag{3.2.6a}$$

Similarly we shall abbreviate the tensor product of ℓ unit radial vectors by

$$N_{B_\ell} \equiv n_{b_1}\ldots n_{b_\ell}. \tag{3.2.6b}$$

In this abbreviated notation equation (3.2.4) says

$$Y^{\ell m}(\theta,\phi) = \mathcal{Y}^{\ell m}_{K_\ell} N_{K_\ell} . \tag{3.2.4'}$$

Thus, capital subscript letters denote sequences of lower-case subscript indices; and the number of indices in a sequence is denoted by a subscript to the capital subscript.

The tensors $\mathcal{Y}^{\ell m}_{K_\ell}$ with $-\ell \leq m \leq +\ell$ serve two roles: First, they generate the spherical harmonics of order ℓ (eq. 3.2.4'). Second, they form a basis for the $(2\ell+1)$-dimensional vector space of STF-ℓ tensors; i.e., any STF-ℓ tensor $\mathcal{F}$ can be expanded as

$$\mathcal{F}_{K_\ell} = \sum_{m=-\ell}^{\ell} F^{\ell m} \mathcal{Y}^{\ell m}_{K_\ell} . \tag{3.2.7a}$$

The tensor components $\mathcal{F}_{K_\ell}$ are real if and only if $F^{\ell\,-m} = (-1)^m F^{\ell m}$. The expansion (3.2.7a) can be inverted as follows (see, e.g., TV)

$$F^{\ell m} = 4\pi \frac{\ell!}{(2\ell+1)!!} \mathcal{F}_{K_\ell} \mathcal{Y}^{\ell m}_{K_\ell} \tag{3.2.7b}$$

where the "double factorial" is defined by

$$n!! \equiv n(n-2)(n-4)\ldots(2 \text{ or } 1). \tag{3.2.8}$$

In practical calculations one can use spherical harmonics and STF-ℓ tensors interchangeably: Consider any sphere centered on the coordinate origin, and on that sphere consider any scalar function $f(\theta,\phi)$. One can expand $f(\theta,\phi)$ in spherical harmonics with complex-number expansion coefficients

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} F^{\ell m} Y^{\ell m}(\theta,\phi); \tag{3.2.9a}$$

alternatively, one can expand it in powers of the unit radial vector $\underset{\sim}{n}$, with coefficients that are STF-ℓ tensors

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty} \mathcal{F}_{K_\ell} N_{K_\ell} . \tag{3.2.9b}$$

The expansion coefficients of the two schemes are related by equations (3.2.7a,b).

Similar expansions can be made for vector fields and tensor fields. The resulting formalism, which relates vector and tensor spherical harmonics to STF-ℓ tensors, is presented in TV.

3.3. Multipole Expansion of Gravitational-Wave Field

The vacuum Einstein field equations permit the gravitational-wave amplitude A_{ij} (eq. 3.1.1) to be any symmetric, transverse, traceless function of retarded time $t-r$ and angle. One can show [see, e.g., TV] that this fact allows A_{ij} to be expanded in a multipole series of the form

$$h_{ij}^{TT} = r^{-1}A_{ij}$$

$$= \left[\sum_{\ell=2}^{\infty} (4/\ell!)r^{-1}\,{}^{(\ell)}\mathcal{I}_{ijA_{\ell-2}}(t-r)\,N_{A_{\ell-2}} \right. \qquad (3.3.1)$$

$$\left. + \sum_{\ell=2}^{\infty} [8\ell/(\ell+1)!]r^{-1}\varepsilon_{pq(i}\,{}^{(\ell)}\mathcal{S}_{j)pA_{\ell-2}}(t-r)n_q N_{A_{\ell-2}}\right]^{TT} .$$

Here the normalizations (factors of 4, 8, ℓ, etc.) have been chosen to make equations (4.4.3) look simple; ε_{abc} is the Levi-Civita tensor; parentheses around tensor indices denote symmetrization; ${}^{(\ell)}\mathcal{I}_{A_\ell}(t-r)$ and ${}^{(\ell)}\mathcal{S}_{A_\ell}(t-r)$ are the trace-free, symmetric coefficients of the expansion evaluated at retarded time t-r; a prefix superscript in parentheses, e.g., ${}^{(k)}\mathcal{S}(u)$, means that the quality is to be differentiated k times with respect to its argument u

$$^{(k)}\mathcal{S}(u) \equiv d^k\mathcal{S}/du^k; \qquad (3.3.2)$$

and TT means that the transverse, traceless part is to be taken (eq. 1.3.3).

The coefficients ${}^{(\ell)}\mathcal{I}_{A_\ell}$ and ${}^{(\ell)}\mathcal{S}_{A_\ell}$ of the spherical harmonic expansions--integrated ℓ times (superscript ℓ removed)--are called the "mass" and "current" moments of the radiation field:

$\mathcal{I}_{ab} \equiv$ (mass quadrupole moment), $\mathcal{S}_{ab} \equiv$ (current quadrupole moment),

$\mathcal{I}_{abc} \equiv$ (mass octupole moment), $\mathcal{S}_{abc} \equiv$ (current octupole moment), (3.3.3)

$$\mathcal{I}_{A_\ell} \equiv (\text{mass } \ell\text{-pole moment}), \quad \mathcal{S}_{A_\ell} \equiv (\text{current } \ell\text{-pole moment}).$$

They are trace-free, symmetric tensors which depend on retarded time.

Notice that the mass quadrupole part of the radiation field (3.3.1) has a form familiar from the "Quadrupole Moment Formalism"

$$(h^{TT}_{jk})_{\text{mass quadrupole}} = 2r^{-1}\, \ddot{\mathcal{I}}^{TT}_{jk} \qquad (3.3.4)$$

[cf. eq. (2.6.4)] . Also notice that the mass-multipole moments produce a radiation field with "electric-type parity" (also called "even-type parity"), $\pi = (-1)^{\ell}$; while the current-multipole moments produce a radiation field with "magnetic-type" ("odd-type") parity, $\pi = (-1)^{\ell+1}$

If the radiation field is known, one can project out its multipole moments using the following integrals over a sphere of constant (t-r):

$$^{(\ell)}\mathcal{I}_{B_\ell} = \left[\begin{matrix}\text{Symmetric, trace-}\\ \text{free part of}\end{matrix}\right]\left\{\frac{\ell(\ell-1)(2\ell+1)!!}{2(\ell+1)(\ell+2)}\,\frac{r}{4\pi}\int h^{TT}_{b_1 b_2}\, n_{b_3}\cdots n_{b_\ell}\, d\Omega\right\}, \qquad (3.3.5a)$$

$$^{(\ell)}\mathcal{S}_{B_\ell} = \left[\begin{matrix}\text{Symmetric, trace-}\\ \text{free part of}\end{matrix}\right]\left\{\frac{(\ell-1)(2\ell+1)!!}{4(\ell+2)}\,\frac{r}{4\pi}\int \varepsilon_{b_1 jk}\, n_j\, h^{TT}_{kb_2}\, n_{b_3}\cdots n_{b_\ell}\, d\Omega\right\} \qquad (3.3.5b)$$

Multipole expansions of a generic radiation field have been given in various notations by Sachs (1961), Pirani (1965), Mathews (1962), Janis and Newman (1965), Bonnor and Rotenberg (1966), and Campbell and Morgan (1971). I like the above STF notation -- which is due to Sachs (1961), and Pirani (1965)-- because it ties in nicely with the theory of slow-motion sources (see below) and with the standard form of the quadrupole-moment formalism. Elsewhere (TV) I exhibit the relationship between the above STF expansions and expositions that use other conventions for spherical harmonics.

3.4. Multipole Expansion of Energy, Momentum, and Angular Momentum in Waves

The energy and linear momentum carried off by the radiation field (3.3.1) are most easily evaluated using the Isaacson (1968) stress-energy tensor for gravitational waves

$$T^{GW}_{\alpha\beta} = (1/32\pi)\left\langle h^{TT}_{jk,\alpha}\, h^{TT}_{jk,\beta}\right\rangle \qquad (3.4.1)$$

(cf. MTW, §§35.7 and 35.15). Here the brackets, $\langle\ \rangle$, denote an average over several wavelengths. The power radiated into a unit solid angle about the radial, $\tilde{n}$, direction is

$$(dE/d\Omega dt) \equiv -r^2 n_j T^{GW}_{j0} = +r^2\ T^{GW}_{00}\ ; \qquad (3.4.2)$$

and the total power radiated is the integral of this power over a sphere lying in the local wave zone:

$$dE/dt = \int (dE/d\Omega dt) d\Omega. \qquad (3.4.3)$$

By inserting the gravitational-wave field (3.3.1) into equations (3.4.1)-(3.4.3) and integrating one obtains (see TV)

$$\frac{dE}{dt} = \sum_{\ell=2}^{\infty} \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell} \frac{1}{\ell!(2\ell+1)!!} \left\langle {}^{(\ell+1)}\mathcal{I}_{A_\ell}\ {}^{(\ell+1)}\mathcal{I}_{A_\ell} \right\rangle$$
$$+ \sum_{\ell=2}^{\infty} \frac{4\ell(\ell+2)}{(\ell-1)} \frac{1}{(\ell+1)!(2\ell+1)!!} \left\langle {}^{(\ell+1)}\mathcal{S}_{A_\ell}\ {}^{(\ell+1)}\mathcal{S}_{A_\ell} \right\rangle. \qquad (3.4.4)$$

The waves carry linear momentum out radially--and, as with any locally plane-fronted radiation field, the magnitude of their momentum flux is the same as that of their energy flux:

$$(dP_j/d\Omega dt) \equiv r^2\ n_k\ T^{GW}_{kj} = r^2\ n_j T^{GW}_{00} = n_j\ (dE/d\Omega dt) \qquad (3.4.5)$$

(cf. eq. 35.77j of MTW). The total linear momentum carried off by the waves,

$$dP_j/dt = \int (dP_j/d\Omega dt) d\Omega, \qquad (3.4.6)$$

can be evaluated by inserting the wave field (3.3.1) into the above equations and integrating:

$$\frac{dP_j}{dt} = \sum_{\ell=2}^{\infty} \left\{ \frac{2(\ell+2)(\ell+3)}{\ell(\ell+1)!(2\ell+3)!!} \left\langle {}^{(\ell+2)}\mathcal{I}_{jA_\ell}\ {}^{(\ell+1)}\mathcal{I}_{A_\ell} \right\rangle + \right.$$
$$+ \frac{8(\ell+3)}{(\ell+1)!(2\ell+3)!!} \left\langle {}^{(\ell+2)}\mathcal{S}_{jA_\ell}\ {}^{(\ell+1)}\mathcal{S}_{A_\ell} \right\rangle + \qquad (3.4.7)$$
$$\left. + \frac{8(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!} \left\langle \varepsilon_{jpq}\ {}^{(\ell+1)}\mathcal{I}_{pA_{\ell-1}}\ {}^{(\ell+1)}\mathcal{S}_{qA_{\ell-1}} \right\rangle \right\}.$$

Note, that, whereas the energy radiated (eq. 3.4.4) involves a beating of each multipole moment against itself, the linear momentum radiated (eq. 3.4.7) involves a beating of "adjacent" multipole moments.

The dependence of the wave amplitude, h^{TT}_{ij}, on angle causes its wave fronts to be not quite precisely spherical--and thereby enables the waves to carry off angular momentum. One might hope that the angular momentum loss could be calculated by integrating $\epsilon_{jab}\, x_a\, T^{GW}_{b0}$ over a sphere surrounding the source. Unfortunately, such a procedure fails--for this reason: the averaging process that underlies equation (3.4.1) for $T^{GW}_{\alpha\beta}$ treats as zero the "tiny corrections" which die out as $1/r^3$. However, it is precisely the $1/r^3$ part that carries off the angular momentum. I was vaguely aware of this fact when writing the relevant sections of MTW, but was not sufficiently certain to spell it out explicitly. Subsequently Bryce DeWitt (1971) derived a simple, correct expression for the flux of angular momentum:

$$\frac{dS_j}{dt} = \frac{1}{16\pi}\int \varepsilon_{jpq}\, x_p \left\langle (h^{TT}_{qa}\; {}^{(1)}h^{TT}_{ab})_{,b} - \frac{1}{2} h^{TT}_{ab,q}\; {}^{(1)}h^{TT}_{ab} \right\rangle r^2\, d\Omega. \tag{3.4.8}$$

A word of interpretation is needed. This equation is correct only if the integral is evaluated in the asymptotic rest frame of the source. (Similarly for all previous formulae in this section.) As the source's linear momentum changes (eq. 3.4.7), its asymptotic rest frame gradually changes; and one must gradually change the reference frame in which one evaluates (3.4.8) (and all previous integrals).

Return to equation (3.4.8). By inserting the wave field (3.3.1) and integrating, one obtains

$$\frac{dS_j}{dt} = \sum_{\ell=2}^{\infty} \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!\,(2\ell+1)!!} \left\langle \varepsilon_{jpq}\; {}^{(\ell)}\mathcal{I}_{pA_{\ell-1}}\; {}^{(\ell+1)}\mathcal{I}_{qA_{\ell-1}} \right\rangle$$
$$+ \sum_{\ell=2}^{\infty} \frac{4\ell^2(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!} \left\langle \varepsilon_{jpq}\; {}^{(\ell)}\mathcal{S}_{pA_{\ell-1}}\; {}^{(\ell+1)}\mathcal{S}_{qA_{\ell-1}} \right\rangle . \tag{3.4.9}$$

Notice that this angular momentum radiated, like the energy radiated (eq. 3.4.4), involves a beating of each multipole with itself.

3.5. Discussion of the Multipole Formalism

Of what use are the above formulas? I view them as tools to to be used in studying the generation of gravitational waves from explicit sources. Given a source, one identifies the local wave zone. Using any technique one can dream up (and this is the tough part of the analysis!), one calculates the time-changing multipole moments of the source. One then plugs into the above formulas to get the radiation field and the rate it carries off energy, momentum, and intrinsic angular momentum.

As an alternative application, one can calculate the radiation field of a given source (the tough task; above formulas not necessarily useful); one can use equations (3.3.5) to resolve it into multipole pieces; and one can then use the other formulas above to read off the energy, momentum, and angular momentum radiated.

Once the radiation field h_{ij}^{TT} is known in the local wave zone, one can propagate it on outwards to Earth using the propagation equation of the geometric-optics formalism (§7 below). For typical situations the wave form (3.3.1) will remain highly accurate all the way to Earth, except for uninteresting phase shifts caused by the waves' self energy and by various masses present in the Universe, and except for changes in the weakest of the multipole fields caused by nonlinear beating together of stronger multipole fields.

4. SLOW-MOTION FORMALISMS

4.1. Overview

Turn attention now from weak-field systems (§2) and arbitrary systems (§3) to slow-motion systems--i.e., systems for which

$$\lambda\!\!\!^{-} \equiv (\text{reduced wavelength of radiation}) \gg L \equiv (\text{size of source}). \quad (4.1.1)$$

Thorne (1977b--cited henceforth as TVI) has derived a slow-motion multipole-moment formalism for calculating gravitational-wave generation by such systems. This slow-motion formalism generalizes the quadrupole-moment formalism of §2.6 to include all multipole moments, and to encompass sources with arbitrarily strong internal gravity; cf. Table 1.

The slow-motion formalism takes, as its starting point, a multipole analysis of the external gravitational field of a stationary source. That stationary multipole analysis is described in §4.2; and the slow-motion wave-generation formalism built on it is described in §4.3. Then, in §4.4 the weak-field limit of the slow-motion formalism is described. Throughout § 4 we shall use subscripted coordinate indices: $x_\alpha \equiv x^\alpha$.

4.2. Multipole Moments of a Stationary Source

Consider a stationary system residing in asymptotically flat spacetime and surrounded by vacuum. It is well known[4] that the vacuum geometry of spacetime surrounding such a stationary system is uniquely determined by two families of time-independent multipole moments: the mass moments, and the current moments. The object of this section is to give a definition of these moments (i.e., a set of conventions for them) that will tie in simply with radiation theory. More specifically, if the stationary source is set into very slow motion (§4.3 below), its moments as defined here (§4.2) will be identical to those which characterize the emitted radiation (§3 above).

4.2.1. ACMC-N Coordinate Systems

As a tool in defining the moments of a stationary source, we introduce a special class of coordinate systems: A coordinate system will be called "asymptotically Cartesian and mass centered to order N" ("ACMC-N") if and only if its metric coefficients have the following form. Expand them in inverse powers of "radius"

$$r \equiv [(x_1)^2 + (x_2)^2 + (x_3)^2]^{1/2} \quad . \tag{4.2.1}$$

The coefficients must be time-independent (so the Killing vector is $\partial/\partial x_0$); the leading term must be the Minkowskii metric; and the remaining terms must have the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{\overset{0}{A}_{\alpha\beta}(\underset{\sim}{n})}{r} + \frac{\overset{1}{A}_{\alpha\beta}(\underset{\sim}{n})}{r^2} + \ldots + \frac{\overset{N+1}{A}_{\alpha\beta}(\underset{\sim}{n})}{r^{N+2}} \tag{4.2.2}$$
$$+ \text{[terms that die out faster than } 1/r^{N+2}],$$

where

$$\underset{\sim}{n} = \underset{\sim}{x}/r = (x_j/r)\,(\partial/\partial x_j). \tag{4.2.3}$$

Expand each of the coefficients $\overset{P}{A}_{\alpha\beta}(\underset{\sim}{n})$ in spherical harmonics.

[4]See, e.g., van der Burg (1968), Geroch (1970), Clarke and Sciama (1971).

$A^0_{\alpha\beta}$ and A^1_{00} must involve only harmonics of order 0.

A^1_{0j} and A^1_{jk} must involve only harmonics of order 0 and 1.

(4.2.4)

$A^2_{\alpha\beta}$ must involve only harmonics of order 0,1, and 2.

$A^3_{\alpha\beta}$ must involve only harmonics of order 0,1,2, and 3

.

.

.

(4.2.4 con'd)

$A^N_{\alpha\beta}$ must involve only harmonics of order 0,1,...,N.

Note the absence of a dipole in A^1_{00}; that is what makes the coordinates "mass centered".

That such a coordinate system always exists is proved in TVI by showing that "deDonder coordinates," with appropriate specialization of the gauge, are ACMC-∞.

4.2.2. Multipole Moments

Given an ACMC-N coordinate system, we define the multipole moments of order $\ell \leq N+1$ in the following manner: The ℓ-order part of the $1/r^{\ell+1}$ piece of g_{00} is the mass ℓ-pole moment; and the ℓ-order, "odd-type parity" part of the $1/r^{\ell+1}$ piece of g_{0j} is the current ℓ-pole moment. More specifically,

$$A^\ell_{00} = [2(2\ell-1)!!/\ell!]\, \mathcal{I}_{A_\ell} N_{A_\ell} + (\text{parts of order} < \ell) \qquad (4.2.5a)$$

$$A^\ell_{0j} = [-4\ell(2\ell-1)!!/(\ell+1)!]\, \varepsilon_{jpq}\, \mathcal{S}_{pA_{\ell-1}}\, n_q N_{A_{\ell-1}} \qquad (4.2.5b)$$

$$+ (\text{"even-type" parts of order } \ell) + (\text{parts of order} < \ell)$$

(4.2.5b)

where $\mathcal{I}_{A_\ell}$ is the ℓ-pole mass moment, and $\mathcal{S}_{A_\ell}$ is the ℓ-pole current moment.

In TVI the self-consistency of this definition is proved; i.e., it is shown that every ACMC-N coordinate system gives the same result for the moments of order $\ell \leq N + 1$.

This definition of multipole moments is far less elegant than definitions given by other researchers.[4] However, the other definitions lead to moments which fail to tie in nicely with radiation theory. My moments must be nonlinear, algebraic functions of the moments defined elsewhere,[4] but I have not attempted to exhibit that relationship explicitly.

The above definition of multipole moments is embodied in the following formula for the metric coefficient of an ACMC-N coordinate system:

$$g_{00} = -1 + \frac{2\mathcal{I}}{r} + \frac{(0\text{-pole})}{r^2} + \frac{1}{r^3}[3\mathcal{I}_{ab}n_a n_b + (1 \text{ pole}) + (0 \text{ pole})$$

$$+\ldots+ \frac{1}{r^{\ell+1}}\left[\frac{2(2\ell-1)!!}{\ell!}\mathcal{I}_{A_\ell}N_{A_\ell} + (\ell\text{-1 pole})+\ldots+(0 \text{ pole})\right]$$

$$+\ldots+ \frac{1}{r^{N+1}}\left[\frac{2(2N-1)!!}{N!}\mathcal{I}_{A_N}N_{A_N} + (N\text{-1 pole})+\ldots+(0 \text{ pole})\right]$$

$$+\frac{1}{r^{N+2}}\left[\frac{2(2N+1)!!}{(N+1)!}\mathcal{I}_{A_{N+1}}N_{A_{N+1}} + (\text{poles with } \ell \neq N+1)\right]$$

$$+\left[\text{terms that die out faster than } 1/r^{N+2}\right]. \qquad (4.2.6a)$$

$$g_{0j} = -\frac{1}{r^2}\left[2\varepsilon_{jpq}\mathcal{S}_p n_q + \begin{pmatrix}1 \text{ pole part with}\\ \text{parity } \pi = -\end{pmatrix} + (0 \text{ pole})\right]$$

$$-\ldots- \frac{1}{r^{\ell+1}}\left[\frac{4\ell(2\ell-1)!!}{(\ell+1)!}\varepsilon_{jpq}\mathcal{S}_{pA_{\ell-1}}n_q N_{A_{\ell-1}} + \right.$$

$$\left. + \begin{pmatrix}\ell \text{ pole part with}\\ \text{parity } \pi = (-1)^\ell\end{pmatrix} + (\ell\text{-1 pole}) + \ldots + (0 \text{ pole})\right]$$

$$-\ldots- \frac{1}{r^{N+1}}\left[\frac{4N(2N-1)!!}{(N+1)!}\varepsilon_{jpq}\mathcal{S}_{pA_{N-1}}n_q N_{A_{N-1}} + \right.$$

$$\left. + \begin{pmatrix}N \text{ pole part with}\\ \text{parity } \pi = (-1)^N\end{pmatrix} + (N\text{-1 pole}) + \ldots + (0 \text{ pole})\right] +$$

$$- \frac{1}{r^{N+2}} \left[\frac{4(N+1)(2N+1)!!}{(N+2)!} \varepsilon_{jpq} \mathcal{S}_{pA_N} n_q N_{A_N} + \begin{Bmatrix} \text{N+1 pole part with} \\ \text{parity } \pi = (-1)^{N+1} \end{Bmatrix} \right.$$

$$\left. + (\text{parts with } \ell \neq N+1) \right]$$

$$+ [\text{terms that die out faster than } 1/r^{N+2}] \quad . \qquad (4.2.6b)$$

$$g_{jk} = \delta_{jk} + \frac{(0 \text{ pole})}{r} + \frac{1}{r^2} [(1 \text{ pole}) + (0 \text{ pole})] + \ldots$$

$$+ \frac{1}{r^{\ell+1}} [(\ell \text{ pole}) + \ldots + (0 \text{ pole})] + \frac{1}{r^{N+1}} [(N \text{ pole})$$

$$+ \ldots + (0 \text{ pole})] + \frac{1}{r^{N+2}} \left[\begin{matrix} \text{any angular} \\ \text{dependence} \end{matrix} \right]$$

$$+ [\text{terms that die out faster than } 1/r^{N+2}]. \qquad (4.2.6c)$$

4.2.3. Example: The Kerr Metric

As an example, consider the Kerr metric (external field of a stationary black hole). In Boyer-Lindquist coordinates, made quasi-Cartesian by defining

$$x_1 = r \sin\theta \cos\phi, \; x_2 = r \sin\theta \sin\phi \;, \; x_3 = r \cos\theta,$$

the nonzero metric coefficients are (cf. page 877 of MTW)

$$g_{00} = - \frac{r^2 + a^2 \cos^2\theta - 2Mr}{r^2 + a^2 \cos^2\theta} = -1 + \frac{2M}{r} - \frac{2Ma^2 \cos^2\theta}{r^3} + 0\left(\frac{1}{r^5}\right) ,$$

$$g_{0j} = - \frac{2\varepsilon_{jpq} J_p n_q}{r^2 + a^2 \cos^2\theta} = - \frac{2\varepsilon_{jpq} J_p n_q}{r^2} + 0\left(\frac{1}{r^4}\right) , \qquad (4.2.7)$$

$$g_{jk} = \delta_{jk} + \frac{2M}{r} n_j n_k + \frac{1}{r^2} [(0 \text{ pole}) + (2 \text{ pole})] + 0\left(\frac{1}{r^3}\right),$$

where

$$\underset{\sim}{J} \equiv Ma(\partial/\partial x_3). \qquad (4.2.8)$$

The presence of a quadrupole $1/r^2$ term in g_{jk} prevents this coordinate system from being ACMC-1; it is only ACMC-0. Hence, from the metric coefficients we can read off only the moments of order 0 and 1:

$\mathcal{I} \equiv$ (mass monopole moment; i.e., total mass-energy) = M;

$\mathcal{I}_a \equiv$ (mass dipole moment) = 0 [guaranteed to vanish because coordinates are mass centered];

$\underset{\sim}{\mathcal{S}} \equiv$ (current dipole moment; i.e., total angular momentum)

$$= Ma(\partial/\partial x_3) \quad . \qquad (4.2.9)$$

In order to deduce the higher-order moments one must perform a coordinate transformation that gets rid of the offending quadrupole term in g_{jk}. Such a transformation is exhibited in TVI.

4.3. The Slow-Motion, Multipole-Moment Formalism For Strong-Field Sources

Turn attention now from stationary sources to the slow-motion, multipole-moment formalism.

Consider a specific (but arbitrary) slow-motion, isolated system. If the system were not changing at all ($\lambda\!\!\!^{-} = \infty$), the metric everywhere outside the source would be stationary -- i.e., it would be describable by the stationary multipole-moment formalism of § 4.2. We can pass from that stationary metric to the true metric by gradually turning on the time dependence of the system's multipole moments -- i.e., by gradually letting $\lambda\!\!\!^{-}$ decrease from infinity. Mathematically this is achieved by expanding the metric simultaneously in the two small dimensionless parameters R/r and $R/\lambda\!\!\!^{-}$, where R is the length scale that characterizes the weak-field, near-zone metric:

$$R \equiv \underset{\ell=1,2,\ldots}{\text{Maximum}} \left\{ \left|\frac{\mathcal{I}_{A_\ell}}{M}\right|^{1/\ell} , \left|\frac{\mathcal{S}_{A_\ell}}{M}\right|^{1/\ell} \right\} \qquad (4.3.1)$$

See TVI for details. For typical (not highly spherical) sources, R will be approximately equal to the size of the source L.

At zero order in $R/\lambda\!\!\!^{-}$, the R/r expansion of the metric must be formally identical to the general stationary expansion of § 4.2. But the time dependence of that "zero-order solution" will generate corrections or order $(R/\lambdabar)$, $(R/\lambdabar)^2$, ... In TVI it is shown that those corrections are determined uniquely by the (time dependent) multipole moments, plus the demand that the near-zone metric match onto outoing waves. TVI also performs the match onto outgoing waves in the local wave zone, with the following result: The matching requires that the moments $\mathcal{I}_{A_\ell}$, $\mathcal{S}_{A_\ell}$ that appear in the zero-order [i.e., $(R/\lambdabar)^0$] near-zone solutions [eqs. (4.2.6)] be identical to the moments that appear in the radiation field [eq. (3.3.1)] -- except for differences that are typically negligible:

$$|\delta \mathcal{I}_{A_\ell}| \lesssim MR^\ell (M/\lambdabar), \quad |\delta \mathcal{S}_{A_\ell}| \lesssim MR^\ell (M/\lambdabar). \tag{4.3.2a}$$

For the most strongly radiating moment $\mathcal{M}_{A_\ell}$ (usually the mass quadrupole moment $\mathcal{I}_{A_2}$) the disagreement is

$$|\delta \mathcal{M}_{A_\ell}| \lesssim |\mathcal{M}_{A_\ell}| (M/\lambdabar). \tag{4.3.2b}$$

This result leads to the following slow-motion formalism for calculating the generation of gravitational waves: (1) Analyze the structure and evolution of the system in any convenient coordinates and by any fairly accurate approximation scheme. (2) From that analysis obtain an approximation to the external gravitational field which, at any instant, satisfies (to some degree of accuracy) the time-independent, vacuum Einstein field equations. (3) By transforming that external field to an ACMC coordinate system, read off its dominant multipole moments (the moments with the largest values of ${}^{(\ell)}\mathcal{I}_{A_\ell}$ or ${}^{(\ell)}\mathcal{S}_{A_\ell}$). (4) Plug those dominant moments into the gravitational-wave formulae of §3.

One attractive approximation scheme for use in steps (1) and (2) is the "instantaneous-gravity" approximation. In this scheme one sets to zero all time derivatives of the metric (but) not of the matter variables) when solving the Einstein field equations. This has the effect of removing all dynamical freedom from the gravitational field and making gravitational interactions within the source instantaneous rather than retarded. One automatically obtains an external gravitational field which satisfies the time-independent vacuum field equations; and, unless one has made a foolish choice of coordinates, the moments which one computes from that external field should contain errors no larger than $L/\lambdabar$. The radiation field computed from those moments will then have fractional errors $L/\lambdabar$ and $M/\lambdabar$. (Here L is the size of the source).

Typically the ℓ-pole mass and current moments will have magnitudes

$$\mathcal{I}_{A_\ell} \sim ML^\ell \ , \ \mathcal{S}_{A_\ell} \sim M(L/\lambdabar)L^\ell \ . \tag{4.3.3}$$

In this case their contributions to the radiation field [eq.(3.3.1)] will be

$$(h^{TT}_{jk})_{\text{mass } \ell\text{-pole}} \sim (M/r)\ (L/\lambdabar)^\ell , \tag{4.3.4a}$$

$$(h^{TT}_{jk})_{\text{current } \ell\text{-pole}} \sim (M/r)\ (L/\lambdabar)^{\ell+1} \ ; \tag{4.3.4b}$$

Since $L/\lambdabar \ll 1$ (slow-motion assumption), mass quadrupole radiation is usually far larger than the other multipoles. The current quadrupole field and mass octupole field are normally a factor $L/\lambdabar$ smaller. It is this fact which allows one to compute only the lowest few multipole moments when applying the slow-motion formalism.

In special cases (e.g., torsional oscillations of a neutron star) the mass quadrupole moment will vanish, or will be far smaller than its normal value. Then the current quadrupole or mass octupole radiation may dominate -- unless they, too, are abnormally small, allowing higher moments to make themselves felt.

Notice that, if one is interested in the linear momentum radiated by the source (eq. 3.4.7), one must compute not just the lowest significant multipole moment, but also the moments "adjacent" to it. Linear momentum is carried off only through the interference of adjacent multipole fields with each other.

4.3.1. A Sample Application

As a typical application of the slow-motion, multipole-moment formalism, consider a slowly rotating neutron star which is not quite axially symmetric. Ipser (1971) has formulated a general relativistic analysis of the interior of such a star, using the Regge-Wheeler (1957) formalism for small, strong-field deviations from spherical symmetry. Ipser's analysis shows how internal stresses, supported by the crystal structure of the star's mantle, maintain the star's deformation. It also gives formulas for the star's near-zone multipole moments in terms of the star's internal structure. By inserting those near-zone multipole moments into the slow-motion wave-generation formalism, one obtains the gravitational-wave field h^{TT}_{jk} produced by the star's rotation, and also the energy and angular momentum radiated. (Ipser computed the gravitational-wave field from his near-zone multipole moments by brute force, and discovered the then surprising result that it had the same form as in weak-field, slow-motion theory. That discovery was the original motivation for my constructing the above formalism.)

4.4 Weak-Field Limit of the Slow-Motion Formalism

For a slow-motion source ($L/\lambda\!\!\!^{-} \ll 1$) with weak internal fields ($\varepsilon \ll 1$) and weak internal stresses ($S^2 \ll 1$), one can use Newtonian theory to analyze the interior region; cf. §2.6. Using Newtonian theory, one can express the multipole moments

$$\mathcal{I}_{A_\ell} \text{ and } \mathcal{S}_{A_\ell}$$

in terms of volume integrals over the source [see, eg., TV for detailed proof].

The multipole volume integrals must be performed in a mass-centered Cartesian coordinate system--i.e., in Cartesian coordinates with

$$\mathcal{I}_j(t) \equiv \int \rho(\underset{\sim}{x},t)\, x_j\, d^3x = 0. \tag{4.4.1}$$

[Note that such a coordinate system is automatically the rest frame of the source, since the time-derivative of equation (4.4.1) can be put in the form

$$\begin{aligned} 0 = d\,\mathcal{I}_j(t)/dt &= \int [\partial\rho(\underset{\sim}{x},t)/\partial t]\, x_j\, d^3x \\ &= \int \rho(\underset{\sim}{x},t)\, v_j d^3x = (\text{momentum of source}). \end{aligned} \tag{4.4.2}$$

The third equality follows from the equation of mass conservation (2.6.3a) and an integration by parts.]

In a mass-centered Cartesian coordinate system, the volume integrals for $\mathcal{I}_{A_\ell}$ and $\mathcal{S}_{A_\ell}$ are

$$\mathcal{I}_{A_\ell} = \text{Symmetric trace-free part of } I_{A_\ell}, \tag{4.4.3a}$$

$$I_{A_\ell} = \begin{pmatrix} \ell\text{'th moment of} \\ \text{mass distribution} \end{pmatrix} = \int \rho x_{a_1} \cdots x_{a_\ell}\, d^3x; \tag{4.4.3b}$$

$$\mathcal{S}_{A_\ell} = \text{Symmetric trace-free part of } S_{A_\ell}, \tag{4.4.4b}$$

$$\begin{aligned} S_{A_\ell} &= \begin{pmatrix} \ell-1 \text{ moment of angular} \\ \text{momentum distribution} \end{pmatrix} \\ &= \int (\varepsilon_{a_1 jk} x_j \rho\, v_k)\, x_{a_2} \cdots x_{a_\ell}\, d^3x. \end{aligned} \tag{4.4.4b}$$

The "symmetric, trace-free part" can be computed from equations (3.2.1) and (3.2.2).

The volume integrals (4.4.3) and (4.4.4) are not precisely equal to the exact near-zone multipole moments of the source, of course. They contain errors of post-Newtonian order (cf. eq. 2.6.1):

$$(\delta \mathcal{I}_{A_\ell}) \text{ due to weak-field assumption} \sim ML^\ell[\varepsilon+(L/\lambda\!\!\!^-)^2+S^2], \tag{4.4.5a}$$

$$(\delta \mathcal{S}_{A_\ell}) \text{ due to weak-field assumption} \sim M(L/\lambda\!\!\!^-)L^\ell[\varepsilon+(L/\lambda\!\!\!^-)^2+S^2]. \tag{4.4.5b}$$

These errors are in addition to the amounts (eqs. 4.3.2) by which the wave-zone multipole moments fail to agree with the near-zone multipole moments.

The weak-field limit of the slow-motion wave-generation formalism can be summarized by the following set of rules: (1) Use the Newtonian theory of gravity to analyze the structure and evolution of the source. (2) Calculate the lowest few multipole moments by evaluating expressions (4.4.3) and (4.4.4) in the mass-centered, Cartesian coordinates. (3) Insert those multipole moments into the local-wave-zone formulas of §3. Those formulas will then describe the lowest few multipoles of the radiation field. (4) Check that the radiation field is larger than the errors inherent (a) in the matching of near zone onto local wave zone (eqs. 4.3.2. with $R=L$), and (b) in the weak-field assumption

$$(\delta h^{TT}_{jk})_{\text{mass } \ell\text{-pole, errors}} \sim (M/r)\ (L/\lambda\!\!\!^-)^\ell[\varepsilon+(L/\lambda\!\!\!^-)^2+S^2], \tag{4.4.6a}$$

$$(\delta h^{TT}_{jk})_{\text{current } \ell\text{-pole, errors}} \sim (M/r)(L/\lambda\!\!\!^-)^{\ell+1}[\varepsilon+(L/\lambda\!\!\!^-)^2+S^2]. \tag{4.4.6b}$$

This weak-field, slow-motion formalism dates back to Einstein (1918), for the mass quadrupole part. The mass octupole and current quadrupole parts were first derived (so far as I know) by Papapetrou (1962, 1971); and the full formalism (in different notations from this) was first derived by Mathews (1962); see also Campbell and Morgan (1971). The earliest versions of the formalism assumed gravity so weak that one had to use Linearized Theory rather than Newtonian Theory in analyzing the source ("no self-gravity"). However, it was soon realized that a modest amount of self-gravity causes no problems in the formalism.

4.5 Post-Newtonian Multipole Formalism

We now return to the Epstein-Wagoner (1975) Post-Newtonian Wave-Generation Formalism, which was described qualitatively in §2.7. In this formalism Epstein and Wagoner find it most convenient to analyze the structure and motion of their source using a different Post-Newtonian gauge from that of Chandrasekhar (1965). Like Chandrasekhar, they describe the matter of their source by perfect-fluid thermodynamic variables measured in its local rest frame

$$\rho_o = \text{mass density}, \quad P = \text{pressure}, \quad \Pi = \text{specific internal energy}; \tag{4.5.1a}$$

and by the coordinate velocity of the fluid

$$v_j \equiv \frac{dx^j}{dt}. \tag{4.5.1b}$$

The internal gravity of the source they describe by the potentials U, V_j, Ψ, and χ which satisfy

$$U_{,jj} = -4\pi\rho_o, \quad V_{j,kk} = -4\pi\rho_o v_j,$$

$$\Psi_{,jj} = -4\pi\rho_o(v^2+U+\tfrac{1}{2}\Pi + \tfrac{3}{2}P/\rho_o), \quad \chi_{,jj} = -2U, \tag{4.5.2}$$

and which are related to the post-Newtonian metric by

$$g_{00} = -1+2U-2U^2+4\Psi-\chi_{,00}+O(\varepsilon^6) \tag{4.5.3a}$$

$$g_{0j} = -4V_j+O(\varepsilon^5) \tag{4.5.3b}$$

$$g_{jk} = 2U\delta_{jk}+O(\varepsilon^4) \tag{4.5.3c}$$

[In the Post-Newtonian formalism one assumes $\varepsilon \sim S^2 \sim (L/\lambda\!\!\!^-)^2$.] The equations governing the evolution of the source in the Epstein-Wagoner gauge are the equation of mass conservation

$$[\rho_o(1+\tfrac{1}{2}\underset{\sim}{v}^2+3U)]_{,0}+[\rho_o(1+\tfrac{1}{2}\underset{\sim}{v}^2+3U)v_j]_{,j}=0, \tag{4.5.4a}$$

the equation of state

$$P=P(\rho_o,\Pi), \tag{4.5.4b}$$

the adiabatic equation of energy conservation

$$\rho_o\, d\Pi/dt + Pv_{j,j}=0, \quad d/dt \equiv \partial/\partial t + v_j\, \partial/\partial x_j, \tag{4.5.4c}$$

and the Euler equations of motion

$$\rho_o dv_i/dt - \rho_o U_{,i} + P_{,i} - (\Pi + \underset{\sim}{v}^2 + 4U + P/\rho_o)P_{,i}$$

$$+ (3\rho_o U_{,0} + P_{,0} + 4\rho_o v_j U_{,j})v_i + (4U - \underset{\sim}{v}^2)\rho_o U_{,i} - 4\rho_o V_{i,0}$$

$$+ 4\rho_o v_k (V_{k,i} - V_{i,k}) + \rho_o (\tfrac{1}{2}\chi_{,00i} - 2\Psi_{,i}) = 0 .$$

After one has computed the structure and evolution of the source using these equations, one can then compute the multipole moments which govern the radiation field. For typical sources -- to which Epstein and Wagoner restrict their attention -- the multipole components of the radiation field have the magnitudes (4.3.4). The mass quadrupole dominates with magnitude $(M/r)(L/\lambda\!\!\!^{-})^2$ and the post-Newtonian formalism is able to compute it with a fractional error $\sim \varepsilon^2 \sim (L/\lambda\!\!\!^{-})^4$. If one wishes to compute the other multipole contributions to similar absolute accuracy, then one must have fractional errors no larger than the following in the various multipole moments (cf. eq. 4.3.4):

$$\mathcal{I}_{A_2}: (L/\lambda\!\!\!^{-})^4, \quad \mathcal{I}_{A_3}: (L/\lambda\!\!\!^{-})^3, \quad \mathcal{I}_{A_4}: (L/\lambda\!\!\!^{-})^2, \quad \mathcal{I}_{A_5}: (L/\lambda\!\!\!^{-}),$$

$$\mathcal{S}_{A_2}: (L/\lambda\!\!\!^{-})^3 \quad \mathcal{S}_{A_3}: (L/\lambda\!\!\!^{-})^2 \quad \mathcal{S}_{A_4}: (L/\lambda\!\!\!^{-}). \qquad (4.5.5)$$

Up to this accuracy the multipole moments can be expressed in the following form:

$$(\mathcal{I}_{jk}, \mathcal{I}_{ijk}, \mathcal{S}_{jk}) = \text{Symmetric trace-free parts of } (I_{jk}, I_{ijk}, S_{jk}),$$

$$I_{jk} = \int\left[\tau_{00}x_j x_k + \frac{11}{21}\tau_{jk}r^2 + \frac{4}{21}\tau_{pp}x_j x_k - \frac{4}{7}x_p\tau_{pj}x_k\right]d^3x, \qquad (4.5.6)$$

$$I_{ijk} = \int\left[\tau_{00}x_i x_j x_k + x_i\tau_{jk}r^2 + \frac{1}{3}\tau_{pp}x_i x_j x_k - x_p\tau_{pi}x_j x_k\right]d^3x,$$

$$S_{ij} = \int\left[x_i\varepsilon_{jpq}x_p\tau^0{}_q - \frac{3}{28}\varepsilon_{ipq}x_p r^2\partial_t\tau_{qj} + \frac{1}{28}x_i\varepsilon_{jpq}x_p\partial_t\tau_{qs}x_s\right]d^3x,$$

$(\mathcal{S}_{ijk}, \mathcal{I}_{ijk\ell})$ equal the Newtonian expressions (4.4.3), (4.4.4).

Here, the "effective energy density". τ_{00}, "effective momentum density" $\tau^0{}_j$, and "effective stress" τ_{jk} are

$$\tau_{00} = \rho_o(1 + \Pi + \underset{\sim}{v}^2 + 4U) - \frac{3}{8\pi}U_{,j}U_{,j} \qquad (4.5.7a)$$

$$\tau^0{}_j = \rho_o(1+\Pi+\underset{\sim}{v}^2+4U)v_j+Pv_j + \frac{3}{4\pi}U_{,0}U_{,j} + \frac{1}{2\pi}U_{,k}\dot{V}_{k,j}$$

$$- \frac{1}{2\pi}V_k U_{,kj} - 2\rho_o V_j, \tag{4.5.7b}$$

$$\tau_{jk} = \rho_o v_j v_k - \frac{1}{4\pi} U_{,j}U_{,k} - \frac{1}{2\pi} UU_{,jk}$$

$$+ \delta_{jk}(P + \frac{3}{8\pi}U_{,i}U_{,i} - 2\rho_o U). \tag{4.5.7c}$$

Epstein and Wagoner use a different notation than ours for the multipole moments. The above formulas for the moments (eqs. 4.5.6) are derived in TV.

5. PERTURBATION FORMALISMS

Thus far I have described two large classes of wave-generation formalisms: weak-field formalisms (§2) and slow-motion formalisms (§4). Now I turn attention to a third large class: Perturbation formalisms.

The fundamental assumption underlying all perturbation formalisms is this: that the entire wave-generation region of space-time can be treated as a small perturbation which radiates, superimposed on a nonradiative but strongly curved "background" Examples are: (1) Small-amplitude pulsations of fully relativistic stars. Here the background is an unperturbed, equilibrium stellar model; and the perturbation is the pulsation. (2) Slow rotation of a slightly nonspherical neutron star (pulsar). Here the background is a nonrotating, spherical star; and the perturbation is both the deformation and the rotation. (3) Motion of a small object in the gravitational field of a black hole. Here the background is the Kerr metric of the black hole; and the small perturbation is the stress-energy tensor of the object, plus the gravitational field it produces.

There are a variety of different perturbation formalisms, each designed to handle a specific type of problem. Some applications make use of several formalisms combined together--and many applications combine a perturbation formalism for the wave-generation region with a multipole analysis of the radiation field (i.e., with a variant of the formalism described in §3).

In a recent review article (§II.C of Thorne 1977c) I have described most of the perturbation formalisms with which I am familiar. Rather than repeat that material here, I simply refer the reader to it.

6. WAVE GENERATION BY SOURCES WITH STRONG INTERNAL FIELDS AND FAST, LARGE-AMPLITUDE MOTIONS

The strongest sources of gravitational waves in the Universe should be sources with strong internal fields ($\varepsilon \sim 1$) and fast ($v \sim L/\lambdabar \sim 1$), large-amplitude internal motions. Examples are the highly nonspherical collapse of a star to form a black hole (or naked singularity, or whatever it does form), and a collision between two black holes. Unfortunately, such sources have eluded all efforts at analytic analysis.[5] There exists no formalism today by which one can calculate the waves they generate. Obviously, an accurate analysis of such systems is the most important and most difficult task lying ahead of us in the theory of gravitational-wave generation.

Fortunately great progress has been made on this task recently by Bryce DeWitt, Larry Smarr, Kenneth Eppley, and others (see Smarr 1977 for a review). Abandoning all hope of a truly analytic analysis, they turn to massive electronic computers as their key tool. Their method is elegant numerical solution of the full, nonlinear Einstein field equations. By now they have encountered and surmounted a number of serious numerical problems. The resulting numerical methods are nearly good enough to give reliable results for strong-field, high-speed, large-amplitude sources--and we can expect true reliability within another year or two. This, when it is achieved, will be very useful in planning gravitational-wave-detection efforts.

[5]An exception is the special situation of a collision between two black holes with relative velocity very nearly the speed of light [$\gamma=(1-v^2)^{-1/2} \gg 1$], for which D'Eath (1977) has formulated a remarkable "colliding-plane-wave" approximation that yields the dominant features of this radiation.

7. PROPAGATION OF WAVES TO EARTH

7.1. The Geometric-Optics Formalism

Once the gravitational-wave field h^{TT}_{jk} is known in the local wave zone, one can then propagate it out through the surrounding Universe to Earth, using the vacuum Einstein field equations. In nearly all situations one can use the geometric-optics approximation to the Einstein equations (e.g. Exercise 35.15 of MTW). In this section I describe the geometric optics formalism in a language different from, but equivalent to MTW.

This formalism, which is valid for $r >> r_I$ (wave zone), describes spacetime by a background metric $g^{(B)}_{\mu\nu}$ through which the waves propagate. The waves are described by a gravitational-wave field $\psi_{\mu\nu}$ which reduces to

$$\psi_{jk} = h^{TT}_{jk}, \quad \psi_{j0} = 0, \quad \psi_{00} = 0 \quad \text{in local wave zone and in asymptotic rest frame of source.} \tag{7.1.1}$$

In an appropriate coordinate system (gauge) the full metric of spacetime is

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + \psi_{\mu\nu} + 0\,(\psi^2). \tag{7.1.2}$$

The waves are distinguished from the background by the very small length scale $\lambdabar$ on which they vary

$$\lambdabar \equiv \begin{Bmatrix}\text{length scale of}\\ \text{variations in } \psi_{\mu\nu}\end{Bmatrix} << \mathcal{R}_B \equiv \begin{Bmatrix}\text{radius of curvature}\\ \text{of background spacetime}\end{Bmatrix}. \tag{7.1.3}$$

The geometric optics formalism remains valid so long as the propagating waves do not encounter regions of extremely strong curvature--i.e. regions where $\mathcal{R}_B \lesssim \lambdabar$.

In applying the geometric optics formalism to a specific problem, one proceeds as follows: (i) In the local wave zone and in the asymptotic rest frame of the source one describes the background metric by the line element

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + 0(M/r)dx^\alpha dx^\beta, \tag{7.1.4}$$

with the source located at the origin. (ii) One constructs null geodesics of $g^{(B)}_{\mu\nu}$ extending radially away from the source. Near the source each geodesic has the form

$$t - r = \tau_e = \text{const.}, \quad \theta = \text{const.}, \quad \phi = \text{const.} \quad \text{in local wave zone.} \tag{7.1.5}$$

These geodesics are called "rays." The gravitational waves propagate along them. (iii) Each ray is labeled by the proper time ("emission time" or "retarded time") τ_e at which it intersects the source; and by this ray labeling τ_e becomes a scalar field extending through all spacetime. (iv) Each ray has an affine parameter ζ, a world line $\mathcal{P}(\zeta)$ with coordinates $x^\alpha(\zeta)$, and a tangent vector ("propagation vector")

$$\vec{k} = d/d\zeta, \qquad k^\alpha = dx^\alpha/d\zeta. \tag{7.1.6}$$

Of course, since $\mathcal{P}(\zeta)$ is a null geodesic, $\vec{k}$ must satisfy

$$\vec{k}^2 \equiv k_\alpha k^\alpha = 0, \qquad k_{\alpha|\beta} k^\beta = 0. \tag{7.1.7}$$

Here and below a slash denotes covariant derivative with respect to the background metric. (v) One normalizes the affine parameter of each ray so that near the source

$$\zeta = r = \left\{\begin{array}{l}\text{proper distance from source}\\ \text{as measured in asymptotic}\\ \text{rest frame; eq.(7.1.4)}\end{array}\right\} \quad \text{in local wave zone.} \tag{7.1.8a}$$

As a result

$$k^0 = 1, \qquad \underset{\sim}{k} = \underset{\sim}{n} = \left\{\begin{array}{l}\text{unit radial vector}\\ \text{pointing away from}\\ \text{source}\end{array}\right\} \quad \begin{array}{l}\text{in source's asymptotic}\\ \text{rest frame and local}\\ \text{wave zone.}\end{array} \tag{7.1.8b}$$

(vi) From the above definitions and constructions one can show that throughout spacetime the gradient of the retarded time is equal to the propagation vector, except for sign

$$\vec{k} = -\vec{\nabla}\tau_e; \qquad k_\alpha = -\tau_{e,\alpha}. \tag{7.1.9}$$

(vii) In the local wave zone one imposes the starting conditions (7.1.1) on the gravitational-wave field $\psi_{\alpha\beta}$. Note that because h^{TT}_{jk} is transverse and traceless, $\psi_{\alpha\beta}$ is initially trace-free and orthogonal to the propagation vector

$$\psi \equiv \psi^\alpha_\alpha \equiv \psi_{\alpha\beta}\, g^{\alpha\beta}_{(B)} = 0, \; \psi_{\alpha\beta} k^\beta = 0. \tag{7.1.10}$$

These properties are preserved as the waves propagate (cf. eq. 7.1.12 below). (viii) Initially, and after it has propagated, $\psi_{\alpha\beta}$ is a rapidly varying function of retarded time τ_e and in addition is a slowly varying function of location along surfaces of constant τ_e:

$$\psi_{\alpha\beta} = \psi_{\alpha\beta}\,(\tau_e; x^0, x^1, x^2, x^3), \tag{7.1.11a}$$

$$\dot{\psi}_{\alpha\beta} \equiv \frac{\partial \psi_{\alpha\beta}}{\partial \tau_e} = 0\left[\frac{\psi_{\alpha\beta}}{\lambda\!\!\!^{-}}\right], \qquad \ddot{\psi}_{\alpha\beta} \equiv \frac{\partial^2 \psi_{\alpha\beta}}{\partial \tau_e^2} = 0\left[\frac{\psi_{\alpha\beta}}{\lambda\!\!\!^{-2}}\right] \tag{7.1.11b}$$

$$\left(\frac{\partial \psi_{\alpha\beta}}{\partial x^\mu}\right)_{\tau_e} = 0\left[\frac{\psi_{\alpha\beta}}{\zeta} + \frac{\psi_{\alpha\beta}}{\mathcal{R}_B}\right], \tag{7.1.11c}$$

$$\psi_{\alpha\beta|\gamma} = \dot{\psi}_{\alpha\beta} k_\gamma + 0\left[\frac{\psi_{\alpha\beta}}{\zeta} + \frac{\psi_{\alpha\beta}}{\mathcal{R}_B}\right]. \tag{7.1.11d}$$

Here the slash ($\psi_{\alpha\beta|\gamma}$) denotes covariant derivative with respect to the background metric, and 0[X] means "of order X." (ix) $\psi_{\alpha\beta}$ propagates along the rays in accordance with the propagation equation

$$\psi_{\alpha\beta|\mu} k^\mu = -\frac{1}{2} k^\mu{}_{|\mu} \psi_{\alpha\beta}. \tag{7.1.12}$$

(x) One uses equation (7.1.12) to propagate the initial, local-wave-zone field out through the Universe to Earth.

Because, in an appropriate gauge, $\psi_{\alpha\beta}$ is the metric perturbation associated with the waves (eq. 7.1.2), one can use the usual formulas (Chapters 35 and 37 of MTW) to calculate from $\psi_{\alpha\beta}$ whatever properties of the waves one wishes. For example, one can use equations (35.62a,e)of MTW to compute the contribution of the waves to the Riemann curvature tensor of spacetime. By virtue of equations (7.1.11) that contribution is

$$R^{(GW)}_{\alpha\beta\gamma\delta} = \frac{1}{2}(\ddot{\psi}_{\alpha\delta} k_\beta k_\gamma + \ddot{\psi}_{\beta\gamma} k_\alpha k_\delta - \ddot{\psi}_{\beta\delta} k_\alpha k_\gamma - \ddot{\psi}_{\alpha\gamma} k_\beta k_\delta) \tag{7.1.13}$$
$$+ \text{ fractional errors of } 0[\lambda\!\!\!^{-}/\zeta + \lambda\!\!\!^{-}/\mathcal{R}_B].$$

Similarly, one can use equation (35.70) of MTW to compute the Isaacson (1968) stress-energy tensor associated with the waves. It reduces to

$$T^{(GW)}_{\alpha\beta} = \frac{1}{32\pi} \langle \dot{\psi}_{\mu\nu} \dot{\psi}^{\mu\nu} \rangle k_\alpha k_\beta, \tag{7.1.14}$$

where < > means "average over several wavelengths of the waves."

The gravitational-wave field is not fully measurable. Only that part which contributes to the Riemann curvature tensor is measurable; the remainder can be changed at will by gauge transformations (infinitesimal coordinate transformations). The form of the generic gauge transformation which preserves all the above equations is

$$\psi^{NEW}_{\mu\nu} = \psi^{OLD}_{\mu\nu} + \Phi_\mu k_\nu + \Phi_\nu k_\mu \ , \qquad (7.1.15a)$$

where ϕ_μ is a vector field which has the same variability properties as $\psi_{\alpha\beta}$ [eqs. 7.1.11], which is orthogonal to the propagation vector

$$\Phi_\mu k^\mu = 0, \qquad (7.1.15b)$$

and which satisfies the propagation equation

$$\Phi_{\mu|\alpha} k^\alpha = -\frac{1}{2} k^\alpha{}_{|\alpha} \phi_\mu \ . \qquad (7.1.15c)$$

An arbitrary observer with 4-velocity $\vec{u}$, at an arbitrary event in spacetime, may find it convenient to make $\psi_{\mu\nu}$ "spatial, transverse, and traceless" in his own rest frame. He can accomplish this by a gauge transformation of the above form with

$$\Phi_\mu = -\frac{u^\alpha \psi_{\alpha\mu}}{u^\gamma k_\gamma} + \frac{u^\alpha \psi_{\alpha\beta} u^\beta}{2(u^\gamma k_\gamma)^2} \ k_\mu \ . \qquad (7.1.16a)$$

The resulting ("NEW"; "Transverse Traceless") gravitational-wave field $\psi^{TT}_{\mu\nu}$ is related to the original ("OLD") one $\psi_{\mu\nu}$ by

$$\psi^{TT}_{xx} = \psi_{xx} = -\psi_{yy} = -\psi^{TT}_{yy}; \quad \psi^{TT}_{xy} = \psi_{xy} = \psi^{TT}_{yx} \ ; \quad \text{all other } \psi^{TT}_{\alpha\beta} = 0$$

in local rest frame of observer, with Minkowskii coordinates so oriented that $\vec{k} = k^0(\vec{e}_0 + \vec{e}_z)$. (7.1.16b)

In other words, this gauge transformation simply throws away all parts of $\psi_{\alpha\beta}$ except those that are purely spatial and are transverse to the propagation direction; and in the process it preserves the tracelessness of $\psi_{\alpha\beta}$.

7.2 Example of Propagation

As an example of the above formalism, consider a gravitational wave emitted by the gravitational collapse of a 10^6 solar-mass star at a time when the universe was only a few million years old.

Idealize the background metric, through which the waves propagate toward Earth, as a perfectly smooth Friedmann metric (MTW chapter 29)

$$ds^2 = g^{(B)}_{\mu\nu} dx^\mu dx^\nu = a^2[-d\eta^2 + d\chi^2 + \Sigma^2(d\theta^2 + \sin^2\theta d\phi^2)] \qquad (7.2.1a)$$

$$a = a(\eta),\ \Sigma = \begin{cases} \sin\chi & \text{if universe is "closed"} \\ \chi & \text{if universe is "flat"} \\ \sinh\chi & \text{if universe is "open"} \end{cases} ;(7.2.1b)$$

and at each event in the universe introduce the orthonormal frame ("local proper rest frame of universe")

$$\vec{e}_{\hat\eta} = \frac{1}{a}\frac{\partial}{\partial\eta},\ \vec{e}_{\hat\chi} = \frac{1}{a}\frac{\partial}{\partial\chi},\ \vec{e}_{\hat\theta} = \frac{1}{a\Sigma}\frac{\partial}{\partial\theta},\quad e_{\hat\phi} = \frac{1}{a\Sigma\sin\theta}\frac{\partial}{\partial\phi}.$$

(7.2.1c)

Place the supermassive star (source) at the origin of the Friedmann spatial coordinate system, $\chi_s = 0$; and let it emit its gravitational waves during an interval of time $\eta_s \le \eta \le \eta_s + \Delta\eta$. The duration of the burst $[\Delta\tau_e = a(\eta_s)\Delta\eta = (\text{a few seconds})]$ will be very short compared to the age of the universe at the time of emission; and hence "a" will change negligibly during the emission

$$a_s \equiv a(\eta_s) \ggg (da/d\eta)_s\ \Delta\eta. \qquad (7.2.2)$$

The rays, along which the waves propagate, are null geodesics emanating from the star's world line $(\chi_s = 0,\ \eta_s \le \eta \le \eta_s + \Delta\eta)$. By solving the geodesic equation in the Friedmann metric one obtains the following equations for the ray originating at $(\chi = 0,\ \eta = \eta_s + \tau_e/a_s)$ and propagating in the (θ_e, ϕ_e) direction:

$$\chi = \eta - (\eta_s + \tau_e/a_s),\quad \theta = \theta_e,\ \phi = \phi_e, \qquad (7.2.3a)$$

$$k^\eta = k^\chi = \frac{d\eta}{d\zeta} = \frac{d\chi}{d\zeta} = \frac{a_s}{a^2},\quad k^\theta = k^\phi = 0. \qquad (7.2.3b)$$

Note that the "physical components" of the propagation vector (components in the local proper rest frame of the universe [7.2.1c]) are

$$k^{\hat\theta} = k^{\hat\phi} = 0,\qquad k^{\hat\eta} = k^{\hat\chi} = (a_s/a) \text{ at general event}$$

(7.2.4)

$$= 1 \quad \text{at source.}$$

The source's retarded time, expressed as a function of Friedmann coordinates, is

$$\tau_e = a_s\,(\eta - \eta_s - \chi); \qquad (7.2.5)$$

cf. equation (7.2.3a). The gradient of τ_e is the negative of the propagation vector, as required by equation (7.1.9).

In the asymptotic rest frame of the source ($\chi \ll 1$; $\eta_s \lesssim \eta \lesssim \eta_s + \Delta\eta$) the Minkowskii radial and time coordinates are related to Friedmann coordinates and to retarded time by

$$r = a_s\chi, \quad t = a_s(\eta - \eta_s); \quad \tau_e = t - r; \qquad (7.2.6a)$$

and the line element is

$$ds^2 = -\,dt^2 + dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (7.2.6b)$$

$$+\ O(M/r)\,dx^\alpha\, dx^\beta \leftarrow \text{contributions from gravity of source}$$

$$+\ O(r^2/a^2 + t\ d\ \ln a/dt)dx^\alpha dx^\beta \leftarrow \text{cosmological corrections.}$$

The gravitational waves, expressed in a gauge that is "TT" with respect to this asymptotic rest frame, have as their only nonzero components

$$\psi_{\hat\theta\hat\theta} = -\psi_{\hat\phi\hat\phi} \equiv \frac{1}{r}\,A_+(\tau_e,\theta,\phi)$$

in local wave zone.(7.2.7)

$$\psi_{\hat\theta\hat\phi} = \psi_{\hat\phi\hat\theta} \equiv \frac{1}{r}\,A_\times(\tau_e,\theta,\phi)$$

The quantities A_+ and $A_\times$ are amplitudes for the two orthogonal polarization states "+" and "x". They are rapidly varying functions of retarded time $\tau_e = t-r$, and slowly varying functions of θ and ϕ :

$$\dot A \equiv \frac{\partial A}{\partial \tau_e} \sim \frac{A}{\lambda\!\!\!^{-}}, \quad \frac{1}{r}\frac{\partial A}{\partial\theta} \backsim \frac{A}{r}, \quad \frac{1}{r\sin\theta}\frac{\partial A}{\partial\phi} \backsim \frac{A}{r} \qquad (7.2.8)$$

[cf. eq. (7.1.11) and note that in the asymptotic rest frame ζ = (affine parameter) = r].

The gravitational-wave field (7.2.7) propagates out into the Friedmann universe by means of the geometric-optics propagation equation (7.1.12). By solving this equation one discovers that

$$a\Sigma\psi_{\hat{\alpha}\hat{\beta}} = \text{constant along each ray,} \tag{7.2.9}$$

where Σ is the cosmological circumference function defined in equation (7.2.1b). Consequently (cf. eqs. [7.2.1b], [7.2.6a], [7.2.7], and [7.2.5]) the gravitational-wave field at an arbitrary event $(\eta, \chi, \theta, \phi)$ in spacetime is

$$\psi_{\hat{\theta}\hat{\theta}} = -\psi_{\hat{\phi}\hat{\phi}} = \frac{1}{a\Sigma} \quad A_{+}[a_s(\eta-\eta_s-\chi), \theta, \phi] \tag{7.2.10a}$$

$$\psi_{\hat{\theta}\hat{\phi}} = \psi_{\hat{\phi}\hat{\theta}} = \frac{1}{a\Sigma} \; A_{\times}[a_s(\eta-\eta_s-\chi), \theta, \phi]. \tag{7.2.10b}$$

Because the source is far from Earth, when these waves reach Earth they look plane waves. An observer on Earth can interpret them in terms of a local Minkowskii coordinate system with basis vectors

$$\frac{\partial}{\partial t} \equiv \vec{e}_0 = \vec{e}_{\hat{\eta}}, \quad \frac{\partial}{\partial z} \equiv \vec{e}_z = e_{\hat{\chi}}, \quad \frac{\partial}{\partial x} \equiv \vec{e}_x = \vec{e}_{\hat{\theta}}, \quad \frac{\partial}{\partial y} \equiv \vec{e}_y = \vec{e}_{\hat{\phi}}. \tag{7.2.11a}$$

If the location of Earth today is $(\eta_o, \chi_o, \theta_o, \phi_o)$, then the observer's Minkowskii coordinates and the global Friedmann coordinates near Earth are related by

$$t = a_o(\eta-\eta_o), \quad z = a_o(\chi-\chi_o), \quad x = a_o\Sigma_o(\theta-\theta_o), \quad y = a_o\Sigma_o \sin\theta_o(\phi-\phi_o). \tag{7.2.11b}$$

The observer on Earth describes the universe's cosmological structure in terms of a Hubble expansion rate H_o and a deceleration parameter q_o; and he describes the source of the waves as having a cosmological redshift Z_s, which is related to the expansion factor by

$$1 + Z_s = a_o/a_s. \tag{7.2.12}$$

The equations of Friedmann cosmology permit one to express $a_o\Sigma_o \equiv$ (circumference of a circle that passes through Earth and is centered on the source at redshift Z_s)/2π in terms of H_o, q_o, and Z_s:

$$a_o\Sigma_o \equiv R = \frac{H_o^{-1}}{q_o^{\,2}(1+Z_s)}\left[1-q_o+q_oZ_s-(1-q_o)(2q_oZ_s+1)^{1/2}\right]; \tag{7.2.13}$$

cf. eq. (29.33) of MTW. By combining equations (7.2.10) -(7.2.13) one obtains the following expressions for the gravitational-wave field that sweeps past Earth:

$$\psi_{xx} = -\psi_{yy} = \frac{1}{R}A_+\left[\frac{t-z}{1+Z_s} + \text{const.},\ \theta_o,\phi_o\right], \tag{7.2.14a}$$

$$\psi_{xy} = \psi_{yx} = \frac{1}{R}A_\times\left[\frac{t-z}{1+Z_s} + \text{const.},\theta_o,\phi_o\right], \tag{7.2.14b}$$

all other $\psi_{\mu\nu}$ vanish.

Notice that the time dependence, $A[(t-z)/(1+Z_s)+\text{const.}, \theta_o,\phi_o]$, of the waves as they sweep past Earth is identical to the time dependence of the emitted waves as measured by the source, $A[\tau_e,\theta_o,\phi_o]$, except for a redshift of $1+Z_s$ -- the same redshift as one sees in electromagnetic spectral lines. Notice also that for very large redshifts, $Z_s \gg 1$, the amplitude of the waves is independent of redshift:

$$\psi \sim \frac{A}{R} = \frac{A}{H_oq_o} \quad \text{for} \quad Z_s \gg 1 \quad \text{and} \quad Z_s \gg 1/q_o. \tag{7.2.15}$$

8. CONCLUDING REMARKS

Although it may appear from these lectures that the theory of gravitational-wave generation is a highly sophisticated and complex subject, one should not let this blind one to its gross inadequacies.

The strongest sources of gravitational waves in the universe-- and the most promising sources for ultimate detection -- are those with strong internal gravity and fast large-amplitude internal motions. For them the only reliable technique of analysis is massive computer calculations (§6). All the fancy analytic tools of these lectures are helpless in the face of such sources!

I am indebted to Joseph Weber for his patience, and to Alessandra Exposito and Jim Isenberg for valuable assistance in the preparation of the typed manuscript.

REFERENCES

Bonnor, W.B., and Rotenberg, M.A. 1966, Proc. Roy. Soc. A, 289, 247.

Campbell, W.B. and Morgan, T. 1971, Physica, 53, 264.

Chandrasekhar, S. 1964, Astrophys. J., 140, 417.

Chandrasekhar, S. 1965, Astrophys. J., 142, 1488.

Clarke, C.J.S. and Sciama, D.W. 1971, Gen. Rel. Grav., 2, 331.

Crowley, R.J. and Thorne, K.S., 1977, Astrophys. J., in press; also available as Caltech Orange-Aid Preprint 450.

D'Eath, P. 1977, paper submitted for publication in Phys. Rev.

DeWitt, B.S. 1971, Lectures on General Relativity-Stanford, Fall 1971 (unpublished manuscript).

Eardley, D.M., Lee, D.L., and Lightman, A.P. 1974, Phys. Rev. D, 10, 3308.

Eddington, A.S. 1922, Proc. Roy. Soc. A, 102, 268.

Einstein, A. 1918, Berlin Sitzungsberichte, p. 154.

Epstein, R. 1976, Ph.D. Thesis, Stanford University.

Epstein, R. and Wagoner, R.V., 1975, Astrophys.J., 197, 717.

Feynman, R.P., 1963, Acta Physica Polonica, 24, 697.

Geroch, R. 1970, J. Math. Phys., 11, 2580.

Havas, P. and Goldberg, J.N. 1962, Phys. Rev., 128, 398.

Ipser, J.R. 1971, Astrophys. J., 166, 175.

Isaacson, R.A. 1968, Phys. Rev., 166, 1272.

Janis, A.I. and Newman, E.T. 1965, J. Math. Phys., 6, 902.

Kovács, S.J., Jr. and Thorne, K.S. 1977a, paper submitted to Astrophys. J: also available as Caltech Orange Aid Preprint

Kovács, S.J., Jr. and Thorne, K.S. 1977b, paper in preparation.

Landau, L.D. and Lifshitz, E.M. 1962, The Classical Theory of Fields (Oxford: Pergamon Press).

Mashoon, B. 1973, Astrophys. J., 181, L65.

Mathews, J. 1962, J. Soc. Indust. Appl. Math, 10, 768.

Matzner, R.A. and Nutku, Y. 1974, Proc. Roy. Soc. A, 336, 285.

Misner, C.W., Thorne, K.S., and Wheeler, J.A. 1973, Gravitation (San Francisco, W.H. Freeman and Co.); cited in text as MTW.

Papapetrou, A. 1962, Comptes Rendues, 255, 1578.

Papapetrou, A. 1971, Ann. Inst. Henri Poincare, 14, 79.

Peters, P.C. 1970, Phys. Rev. D, 1, 1559.

Peters, P.C. and Mathews, J. 1963, Phys. Rev., 131, 435.

Pirani, F.A.E. 1965 in A. Trautman, F.A.E. Pirani, and H. Bondi, Lectures on General Relativity (Englewood Cliffs, N.J.: Prentice-Hall).

Press, W.H., 1977, paper in press.

Regge T. and Wheeler, J.A. 1957, Phys. Rev., 108, 1063.

Ruffini, R. and Wheeler, J.A. 1971, in Proceedings of the Conference on Space Physics (Paris: ESRO), p. 45.

Sachs, R. 1961, Proc. Roy. Soc. A, 264, 309, esp. Appendix B.

Shapiro, I.I. 1964, Phys. Rev. Lett., 13, 789.

Smarr, L. 1977, in Proceedings of the Eighth Texas Symposium on Relativistic Astrophysics, to be published in Proceedings of the New York Academy of Sciences.

Thorne, K.S. 1977a, paper to be submitted for publication. Cited in text as TV.

Thorne,, K.S. 1977b, paper to be submitted for publication. Cited in text as TVI.

Thorne, K.S., 1977c , in N. Lebovitz ed., Proceedings of the (Chandrasekhar) Symposium on Theoretical Principles in Astrophysics and Relativity (Chicago: University of Chicago Press).

Thorne, K.S. 1977d, paper submitted to Astrophys. J.

Thorne, K.S. and Kovács, S.J., Jr. 1975, Astrophys. J., 200, 245. Cited in text as TK.

Turner, M. 1977, paper submitted for publication; preprint, Stanford University.

van der Burg, M.G.J. 1968, Proc. Roy. Soc. A, 303, 37.

Wagoner R.V. and Will, C.M. 1976, Astrophys. J., 210, 764.

Wagoner, R.V. 1977, in Proceedings of the Academia Nazionale dei Lincei International Symposium on Experimental Gravitation, Pavia, Italy, September 1976; B. Bertotti ed., in press.

Wheeler, J.A. 1966, Ann. Rev. Astron. Astrophys., 4, 423.