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THE ROTATION AND PULSATION OF GENERAL RELATIVISTIC STELLAR MODELS*

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ABSTRACT

A research group in Southern California is currently analyzing, by means of perturbation theory, the rotation and pulsation of fully relativistic stellar models, including the effects of gravitational waves. In this paper are described the astrophysical motivation for this analysis, and some of the ideas, techniques, and results which have emerged from it thus far.

1. INTRODUCTION

In the last four years a number of developments in astronomy and astrophysics have generated interest in problems on the frontier between general relativity and astrophysics. The discoveries of quasars, of explosions in galactic nuclei, and of X-ray sources, as well as rapid progress in the theories of stellar evolution and of supernovae have given impetus to the study of relativistic stellar structure and of gravitational collapse. At the same time, major improvements in radio source counts and the discovery of the cosmic microwave radiation have generated a revolution and upsurge in research on cosmology. (For references and reviews see Robinson, Schild, and Schücking [1965] ; Harrison, Thorne Wakano, and Wheeler [1965] ; Zel'dovich and Novikov [1964, 1965] ; Wheeler [1966] ; Thorne [1966, 1967]).

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In this paper I shall concentrate attention on a small segment of the exciting new field of general relativistic astrophysics : the theory of the rotation and pulsation of general relativistic stellar models. It is widely believed today that neutron stars are formed by the gravitational collapse of ordinary stars which have burned most of their nuclear fuel. (See, e.g., Colgate and White [1966]). When first created by collapse, a neutron star will be endowed with rather large rotation and pulsation amplitudes : The rotation may be sufficient to deform the star markedly, and the pulsation energy may be several per cent of the rest mass-energy of the star. Such large pulsation and rotation will produce important astrophysical effects, according to rough analyses based on Newtonian theory and on the linear approximation to general relativity :

1) Quadrupole and higher-order pulsations should be damped extremely rapidly by the emission of gravitational waves. Zee and Wheeler [1967], Wheeler [1966], and independently Chau [1967] estimate that a burst of gravitational waves containing about 10^{52} ergs (0.01 of the rest mass of the star) should be emitted at a period of about 10^{-3} seconds and with an exponential decay time of about 1 second. Such a burst of gravitational radiation might be detected by the apparatus of Joseph Weber (1966, 1967) if the star is within about 10^4 light years of the earth.

2) The neutron star would probably have a strong magnetic field (as large as 10^{14} gauss, perhaps). Such a field would couple the collapsed neutron star to the gaseous stellar envelope which is ejected during collapse, and which may be associated with a supernova outburst. By means of this magnetic coupling large amounts of energy could be pumped from the stellar pulsation and rotation into the surrounding nebula, with important, astronomically observable effects. (See e.g. Hoyle, Narlikar, and Wheeler [1964], Finzi [1966], Cameron [1965 a,b]).

3) The rotation would probably stabilize the neutron star against collapse even if its mass is a little above the Landau-Oppenheimer-Volkoff limit for non-rotating neutron stars.

The pulsation and rotation of relativistic stellar models is of importance not only for neutron stars, but also for supermassive stars (stars more massive than 10^4 solar masses). Hoyle and Fowler (1965) and Fowler (1964) have suggested that supermassive stars may be the energy sources for quasars and for explosions in the nuclei of galaxies. Although general relativity has little effect on the structure of most supermassive stars, it has marked effect on their stability (Chandrasekhar 1964, Fowler 1964) : In nonrotating stars more massive than 10^5 solar masses general relativistic effects induce collapse before any nuclear fuel can be burned ; and in rotating stars there is a similar limit of 10^8 solar masses. (For a review see Chapter 6 of Thorne [1967]). In order to evaluate better the plausibility of the supermassive-star model for quasars, one needs to understand better the interactions between pulsation, rotation, and general relativistic effects in supermassive stars.

The astrophysical problems described above, along with other considerations, have motivated two rather different research projects on the rotation and pulsation of relativistic stellar models. The first of these, which is being pursued by Chandrasekhar (1965 a,b,c ; 1967 a,b,c) and independently by Fowler (1964, 1966), uses the *post-Newtonian* approximation to general relativity : General relativistic effects are treated as small perturbations in rotating and pulsating Newtonian stellar models. This type of approach is particularly well adapted to supermassive stars, where relativistic effects on the stellar structure are small. However, it cannot be applied with confidence to neutron stars, where relativistic effects on the structure are as large as 30 percent ; nor to problems where gravitational radiation is important, since there are no gravitational waves in the post-Newtonian approximation.

The other research project on rotation and pulsation is well adapted to neutron stars and to gravitational radiation. This project is being pursued by a group in Southern California consisting of myself and S.M. Chitre at Caltech, James Hartle at the University of California at Santa Barbara, and Alfonso Campolattaro at the University of California at Irvine. We treat pulsation and uniform rotation as small perturbations of a fully relativistic, nonrotating stellar model. The perturbations are analyzed to second order in the angular velocity and to first order in the amplitude of pulsation. In the remainder of this paper I shall describe a few of the ideas, techniques, and results of our analyses.

2. THE PERTURBATION ANALYSIS OF ROTATION AND PULSATION

A – Method of Analysis

Our method of analysis is essentially the same for all types of problems - pure rotation, pure pulsation, or mixed rotation and pulsation : (1) Set the problem up in an arbitrary coordinate system to second order in the rotation and first order in the pulsation. The geometry of spacetime is given by

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + h_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (1)$$

where $h_{\mu\nu}$ is the perturbation from the nonrotating, nonpulsating, equilibrium geometry. The fluid is displaced from its equilibrium position by an amplitude ξ, and the density and pressure are changed from their equilibrium values by amounts $\delta\rho$ and δp. (2) Expand $\delta\rho$ and δp in scalar spherical harmonics, expand ξ in vector spherical harmonics, and expand $h_{\mu\nu}$ in tensor spherical harmonics. None – or only a few – of the harmonics of different orders are coupled to each other by the Einstein field equations. (3) Introduce a particular "gauge" (coordinate system) in which the metric perturbation, $h_{\mu\nu}$, takes a simple form for those spherical harmonics of interest. (4) Insert the resulting metric and stress-energy tensors into a computer program (Thorne and Zimmerman 1967) which computes the analytic forms of the corresponding Einstein field equations $G_\mu^\nu = 8\pi T_\mu^\nu$ and $T_{\mu\ ;\nu}^{\nu} = 0$. (5) Use the resulting field equations to prove analytic theorems about the pulsation and rotation. (6) Numerically integrate the field equations to obtain the structures of particular rotating stellar models ; and to obtain eigenfunctions, pulsation frequencies, gravitational-wave amplitudes, and damping times of the normal modes of pulsation.

Some of the results obtained with this method before July 1967 are described below :

B – Structure of Slowly Rotating Stars

James Hartle (1967) has used this method to derive the equations of structure for slowly and rigidly rotating, fully relativistic stellar models, to second order in the angular velocity, Ω. To first order in Ω the only effect of the rotation is to cause a dragging of inertial frames. Hartle shows that the angular velocity associated with this dragging

$$\omega \equiv g_{t\varphi}/g_{\varphi\varphi} \tag{2}$$

is a function only of radius r ; and that it decreases monotonically from the center of the star to infinity. The maximum value of ω, attained at the star's center, is always less than the angular velocity

$$\Omega \equiv u^\varphi/u^t \tag{3}$$

with which the fluid in the star rotates.

At second order (Ω^2) in the rotation the surfaces of constant density and pressure inside the star are deformed from spheres into spheriods ; the star's gravitational field becomes nonspherical (it picks up a quadrupole moment) ; and the mass and mean radius for a given central density and equation of state are changed from their equilibrium values.

The effects of rotation on particular models for neutron stars and supermassive stars have been calculated numerically by Hartle and Thorne (1968). In Figure 1 are shown the effects of rotation on the masses and mean radii of stars at the endpoint of thermonuclear evolution.

C – Pulsation of Nonrotating Stars

The theory of radial pulsations of nonrotating stars was originally developed by Chandrasekhar (1964). Subsequent developments are summarized in Thorne (1966, 1967). Dipole pulsations have been analyzed very recently by Campolattaro and Thorne (1969). In radial and dipole pulsations no gravitational radiation is emitted, and the external gravitational field of the pulsating star is the spherically symmetric, static Schwarzschild geometry.

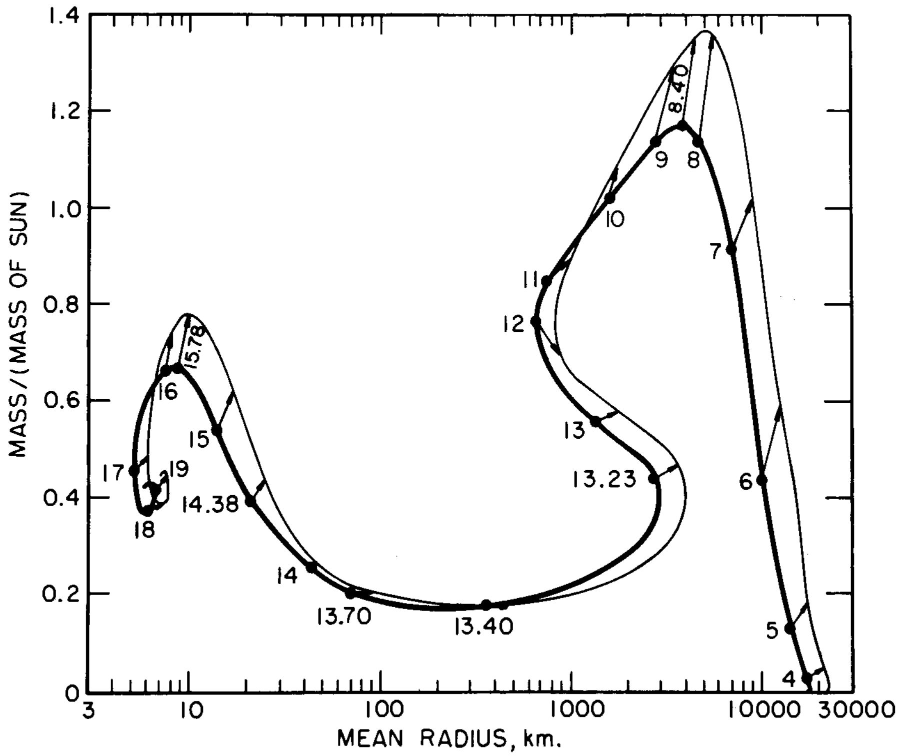

Figure 1. The effect of rotation on some properties of stars at the endpoint of thermonuclear evolution. The thick curve is a plot of mass versus radius, parameterized by logarithm of central density in g/cm^3, for nonrotating stellar models obeying the Harrison-Wheeler equation of state. (See Harrison, Thorne, Wakano, Wheeler 1965). The thin curve is mass versus mean radius ([mean radius] $\equiv$ [Surface area/4π]$^{1/2}$) for stars obeying the same equation of state but rotating with uniform angular velocity $\Omega = (GM/R^3)^{1/2}$. This amount of angular velocity is approximately the amount needed to produce shedding of mass at the star's equator. For smaller angular velocities the deformation of the mass-radius curve is smaller by the dimensionless factor $\Omega^2 R^3/GM$. The small arrows indicate the displacement, with increasing angular velocity, of configurations with given central densities.

Nonradial pulsations of quadrupole and higher order, including the emission of gravitational waves, have been treated analytically by Thorne and Campolattaro (1967) ; but no numerical integrations for particuler stellar models have been attempted yet. Thorne and Campolattaro treat gravitational radiation from non-radial pulsations by a complex-eigenvalue and wave-packet technique patterned after the theory of particle decay in nuclear physics. Attention is concentrated on all normal modes corresponding to a particular spherical harmonic. (There is no coupling between spherical harmonics.) Each such normal mode can be described by an amplitude, $K(r, t)$, for gravitational waves, and by two amplitudes, $W(r, t)$ and $V(r, t)$, for the radial and angular motion of the fluid. Once these three amplitudes are known, the entire gravitational field and fluid motion of the star can be calculated uniquely. Corresponding to *any* complex frequency.

$$\omega = \sigma + i/\tau \qquad \text{(by convention } \sigma \geqslant 0) \ , \tag{4}$$

there is a unique complex set of amplitudes

$$W(r, t) = W_\omega(r) e^{i\omega t} \ , \quad V(r, t) = V_\omega(r) e^{i\omega t} \ , \quad K(r, t) = K_\omega(r) e^{i\omega t} \ , \tag{5}$$

which are solutions to Einstein's field equations. The eigenfunctions, $\{K_\omega(r), W_\omega(r), V_\omega(r)\}$ are determined by an eigenequation of the form

$$\mathcal{L}\{K_\omega, W_\omega, V_\omega\} = \omega^2 \{K_\omega, W_\omega, V_\omega\} \ , \tag{6}$$

where $\mathcal{L}$ is a third order, linear, differential operator in r ; and by certain boundary conditions Far from the star the eigenequation (6) reveals that the gravitational waves consist of an ingoing component with amplitude $C_\omega^{(I)}$, and an outgoing component with amplitude $C_\omega^{(0)}$

$$K_\omega = [C_\omega^{(I)} e^{i\omega(r+2M ln r)} + C_\omega^{(0)} e^{-i\omega(r+2M ln r)}] F(r) \ . \tag{7}$$

Of particular interest are those complex normal modes which have only outgoing gravitational waves ($C_\omega^{(I)} = 0$). For most stars, only a discrete set of normal modes are purely outgoing, and the outgoing modes possess a discrete set of eigenfrequencies : $\omega_1, \omega_2, \ldots$. From the purely outgoing modes – which are always complex ; never real if radiation is present – one can construct real wave packets that exhibit damping by gravitational radiation. For example, if $\omega_n = \sigma_n + i/\tau_n$ is the complex frequency for a purely outgoing normal mode, and if $|i/\tau_n| \ll \sigma_n$, then the real wave packet

$$\{K, U, V\} = \int_0^\infty \frac{\{K_\omega(r), W_\omega(r), V_\omega(r)\} e^{i\omega t} d\omega}{(\omega - \sigma_n)^2 + 1/\tau_n^2} \tag{8}$$

has the form far from the star

$$K(r, t) = \begin{cases} 0 \quad \text{if} \quad 0 < t < r + 2M ln r \\ 2 |\, dC^{(I)}/d\omega \,|_{\omega_n} F(r) e^{-(t-r-2M ln r)/\tau_n} \cos[\sigma_n (t - r - 2M ln r) \\ \qquad + \delta_n] \qquad \text{if} \qquad t > r + 2M ln r. \end{cases} \tag{9}$$

This represents radiation from a star which is in equilibrium before $t \approx 0$, which is set into pulsation at time $t \approx 0$, which pulsates with frequency σ_n thereafter, and which is damped by the emission of gravitational waves with an *e*-folding time ("decay time") τ_n. Notice that, because the gravitational-radiation wave packet has a sharp front and decays exponentially behind that front, it can contain only a finite amount of energy. (The actual amount of energy contained in the packet can be evaluated, after the packet has propagated far from the star into nearly flat space, by means of a stress-energy pseudotensor). Because the packet contains finite total energy, there should be no difficulties with the convergence of higher-order corrections to this linearized perturbation analysis.

The wave packet of equations (8) and (9) was built under the assumption that $|1/\tau_n| \ll \sigma_n$. When this condition is violated, one can still build wave packets which identify σ_n as the pulsation frequency and τ_n as the damping time ; but the analysis near the wave front is much more difficult than when $|1/\tau_n| \ll \sigma_n$.

From the above discussion, it is evident that the characteristic properties of the real, physical, nonradial pulsations of a relativistic star are determined by the complex normal modes with purely outgoing radiation. (See Figure 2). Our research group will begin soon numerical computations of the complex, outgoing normal modes for neutron stars. For a summary of the results of the numerical computations see Thorne (1968).

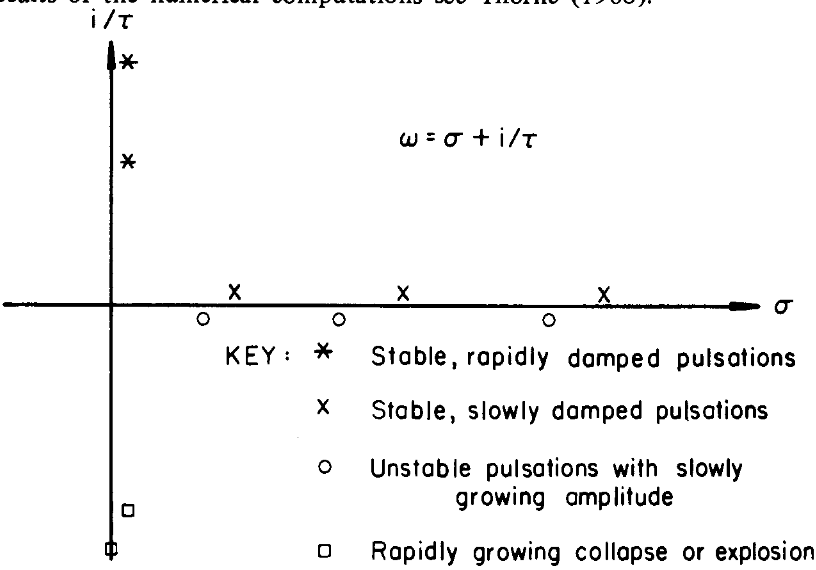

Figure 2. The nature of the physical nonradial pulsations of a relativistic stellar model, as determined by the mathematical complex normal modes with purely outgoing radiation. From equation (9) and the associated discussion, one learns the following : if the eigenfrequencies for the outgoing normal modes lie in the region marked "*", the real pulsations are stable and are damped rapidly by gravitational waves ; if the eigenfrequencies lie in the region marked "x", the real pulsations are stable and are slowly damped ; if the eigenfrequencies lie in the region "0", the real pulsations are unstable but have slowly growing amplitudes — "overstability" in the language of astrophysicists ; if the eigenfrequencies lie in the region "□", the star is unstable against rapidly growing collapse or explosion.

D – **Pulsation of Slowly Rotating Stars**

Chitre, Hartle, and Thorne (1969) are currently studying the effects of rotation on the radial modes of pulsation of relativistic stellar models. No concrete results are available yet from this analysis. However, one expects on general grounds (cf. Wheeler 1966 ; Chau 1967) that the deformation of the star by rotation will mix a small amount of quadrupole pulsation into the radial modes, and will thereby cause gravitational waves to be emitted by the radial modes.

4. CONCLUSION

These are the main features of our work on rotation and pulsation in relativistic stellar models. We hope that, when our project is completed, the results from it will be useful in the design of gravitational radiation detectors, in the further development of gravitational radiation theory, and in the construction or elimination of models for supernovae, supernova remnants, and quasistellar sources.

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