Frame-Dragging Vortexes and Tidal Tendexes Attached to Colliding Black Holes: Visualizing the Curvature of Spacetime

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When one splits spacetime into space plus time, the spacetime curvature (Weyl tensor) gets split into an "electric" part \( \mathcal{E}_{jk} \) that describes tidal gravity and a "magnetic" part \( \mathcal{B}_{jk} \) that describes differential dragging of inertial frames. We introduce tools for visualizing \( \mathcal{B}_{jk} \) (frame-drag vortex lines, their vorticity, and vortexes) and \( \mathcal{E}_{jk} \) (tidal tendex lines, their tendicity, and tendexes) and also visualizations of a black-hole horizon’s (scalar) vorticity and tendicity. We use these tools to elucidate the nonlinear dynamics of curved spacetime in merging black-hole binaries.

**Introduction.**—When one foliates spacetime with space-like hypersurfaces, the Weyl curvature tensor \( C_{\alpha\beta\gamma\delta} \) (the same as Riemann in vacuum) splits into an "electric" and a "magnetic" parts \( \mathcal{E}_{jk} = C_{j\delta\epsilon\kappa} \) and \( \mathcal{B}_{jk} = \frac{1}{2} \epsilon_{jk\ell} C_{\ell\delta\epsilon\kappa} \) (see, e.g., [1] and references therein); both \( \mathcal{E}_{jk} \) and \( \mathcal{B}_{jk} \) are spatial, symmetric, and trace-free. Here the indices are in the reference frame of "orthogonal observers" who move orthogonal to the space slices; \( \mathcal{E}_{jk} \) is their spatial component, \( \epsilon_{jk\ell} \) is their spatial Levi-Civita tensor, and throughout we use units with \( c = G = 1 \).

Because two orthogonal observers separated by a tiny spatial vector \( \xi \) experience a relative tidal acceleration \( \Delta a_j = -\mathcal{E}_{jk} \xi^k \), \( \mathcal{E}_{jk} \) is called the tidal field. And because a gyroscope at the tip of \( \xi \) precesses due to frame dragging with an angular velocity \( \Delta \Omega_j = \mathcal{B}_{jk} \xi^k \) relative to inertial frames at the tail of \( \xi \), we call \( \mathcal{B}_{jk} \) the frame-drag field.

**Vortexes and tendexes in black-hole horizons.**—For a binary black hole, our space slices intersect the three-dimensional (3D) event horizon in a 2D horizon with inward unit normal \( \mathbf{N} \); so \( \mathcal{B}_{\mathbf{NN}} \) is the rate the frame-drag angular velocity around \( \mathbf{N} \) increases as one moves inward through the horizon. Because the connection between rotation and vorticity, we call \( \mathcal{B}_{\mathbf{NN}} \) the horizon’s frame-drag vorticity or simply its vorticity.

Because \( \mathcal{B}_{\mathbf{NN}} \) is boost-invariant along \( \mathbf{N} \) [2], the horizon’s vorticity is independent of how fast the orthogonal observers fall through the horizon and is even unchanged if the observers hover immediately above the horizon (the fiducial observers of the "black-hole membrane paradigm" [3]).

Figure 1 shows snapshots of the horizon for two identical black holes with transverse, oppositely directed spins \( \mathbf{S} \), colliding head-on. Before the collision, each horizon has a negative-vorticity region (red) centered on \( \mathbf{S} \) and a positive-vorticity region (blue) on the other side. We call these regions of concentrated vorticity horizon vortexes. Our numerical simulation [4] shows the four vortexes being transferred to the merged horizon [Fig. 1(b)], then retaining their identities, but sloshing between positive and negative vorticity, and gradually dying, as the hole settles into its final Schwarzschild state; see the movie in Ref. [5].

Because \( \mathcal{E}_{\mathbf{NN}} \) measures the strength of the tidal-stretching acceleration felt by orthogonal observers as they fall through (or hover above) the horizon, we call it the horizon’s tendicity (a word coined by David Nichols from the Latin *tendere* , “to stretch”). On the two ends of the merged horizon in Fig. 1(b), there are regions of strongly enhanced tendicity, called tendexes; cf. Fig. 5 below.

An orthogonal observer falling through the horizon carries an orthonormal tetrad consisting of her 4-velocity \( \mathbf{U} \), the horizon’s inward normal \( \mathbf{N} \), and transverse vectors \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \). In the null tetrad \( \mathbf{I} = (\mathbf{U} - \mathbf{N})/\sqrt{2} \) (tangent to horizon generators), \( \mathbf{n} = (\mathbf{U} + \mathbf{N})/\sqrt{2} \), \( \mathbf{m} = (\mathbf{e}_2 + i\mathbf{e}_3)/\sqrt{2} \), and \( \mathbf{m}^* \), the Newman-Penrose Weyl scalar \( \Psi_2 \) [6], is \( \Psi_2 = (\mathcal{E}_{\mathbf{NN}} + i\mathcal{B}_{\mathbf{NN}})/2 \). Here we use sign conventions of Ref. [7], appropriate for our (− + + −) signature.

Penrose and Rindler [8] define a complex scalar curvature \( \mathcal{K} = \mathcal{R}/4 + iX/4 \) of the 2D horizon, with \( \mathcal{R} \) its

![FIG. 1 (color). Vortexes (with positive vorticity blue, negative vorticity red) on the 2D event horizons of spinning, colliding black holes, just before and just after the merger (from the simulation reported in Ref. [4]).](image)
intrinsic (Ricci) scalar curvature (which characterizes the horizon’s shape) and \( \mathcal{X} \) proportional to the 2D curl of its Hájíček field [9] (the time-space part of the 3D horizon’s extrinsic curvature). Penrose and Rindler show that \( \mathcal{K} = -\Psi_2 + \mu \rho - \lambda \sigma \), where \( \rho \), \( \sigma \), \( \mu \), and \( \lambda \) are spin coefficients related to the expansion and shear of the null vectors \( l \) and \( n \), respectively. In the limit of a shear- and expansion-free horizon [e.g., a quiescent black hole; see Figs. 2(a)–2(c)], \( \mu \rho - \lambda \sigma \) vanishes, so \( \mathcal{K} = -\Psi_2 \), whence \( \mathcal{R} = -2\mathcal{E}_{NN} \) and \( \mathcal{X} = -2\mathcal{B}_{NN} \). As the dimensionless spin parameter \( a/M \) of a quiescent (Kerr) black hole is increased, the scalar curvature \( \mathcal{R} = -2\mathcal{E}_{NN} \) at its poles decreases, becoming negative for \( a/M > \sqrt{3}/2 \); see the blue spots on the poles in Fig. 2(b) compared to solid red spots for the nonrotating hole in Fig. 2(a). In our binary-black-hole simulations, the contributions of the spin coefficients to \( \mathcal{K} \) on the apparent horizons are small [\( L2\text{-norm} \leq 1\% \)] so \( \mathcal{R} \approx -2\mathcal{E}_{NN} \) and \( \mathcal{X} \approx -2\mathcal{B}_{NN} \), except for a time interval \( \sim 5M_{\text{tot}} \) near the merger. Here \( M_{\text{tot}} \) is the binary’s total mass. On the event horizon, the duration of spin-coefficient contributions \( >1\% \) is somewhat longer, but we do not yet have a good measure of it.

Because \( \mathcal{X} \) is the 2D curl of a 2D vector, its integral over the 2D horizon vanishes. Therefore, positive-vorticity regions must be balanced by negative-vorticity regions; it is impossible to have a horizon with just one vortex. By contrast, the Gauss-Bonnet theorem says the integral of \( \mathcal{R} \) over the 2D horizon is \( 8\pi \) (assuming \( S_2 \) topology), which implies the horizon tendicity \( \mathcal{E}_{NN} \) is predominantly negative (because \( \mathcal{E}_{NN} = -\mathcal{R}/2 \) and \( \mathcal{R} \) is predominantly positive). Many black holes have negative horizon tendicity everywhere [an exception is Fig. 2(b)], so their horizon tendexes must be distinguished by deviations of \( \mathcal{E}_{NN} \) from a horizon-averaged value.

**3D vortex and tendex lines.**—The frame-drag field \( \mathcal{B}_{jk} \) is symmetric and trace-free and therefore is fully characterized by its three orthonormal eigenvectors \( \mathbf{e}_j \) and their eigenvalues \( \mathcal{B}_{11}, \mathcal{B}_{22}, \text{and} \mathcal{B}_{33} \). We call the integral curves along \( e_j \) vortex lines, and their eigenvalue \( \mathcal{B}_{jj} \) those lines’ vorticity, and we call a concentration of vortex lines with large vorticity a vortex. For the tidal field \( \mathcal{E}_{jk} \) the analogous quantities are tendex lines, tendicity, and tendexes. For a nonrotating (Schwarzschild) black hole, we show a few tendex lines in Fig. 2(a), and for a rapidly spinning black hole (Kerr metric with \( a/M = 0.95 \)), we show tendex lines in Fig. 2(b) and vortex lines in Fig. 2(c).

If a person’s body (with length \( l \)) is oriented along a positive-tendicity tendex line [blue in Fig. 2(a)], she feels a head-to-foot compressional acceleration \( \Delta a = [\text{tendicity}]\ell \); for negative tendicity (red), it is a stretch. If her body is oriented along a positive-vorticity vortex line [blue in Fig. 2(c)], her head sees a gyroscope at her feet precess clockwise with angular speed \( \Delta \Omega = [\text{vorticity}]\ell \), and her feet see a gyroscope at her head also precess clockwise at the same rate. For negative vorticity (red), the precessions are counterclockwise.

For a nonrotating black hole, the stretching tendex lines are radial, and the squeezing ones lie on spheres [Fig. 2(a)]. When the hole is spun up to \( a/M = 0.95 \) [Fig. 2(b)], its toroidal tendex lines acquire a spiral, and its poloidal tendex lines, when emerging from one polar region, return to the other polar region. For any spinning Kerr hole [e.g., Fig. 2(c)], the vortex lines from each polar region reach around the hole and return to the same region. The red vortex lines from the red north polar region constitute a **clockwise vortex**; the blue ones from the south polar region constitute a **counterclockwise vortex**.

As a dynamical example, consider a Schwarzschild black hole’s fundamental odd-parity \( l = m = 2 \) quasinormal mode of pulsation, which is governed by the Regge-Wheeler perturbation theory [10] and has angular eigenfrequency \( \omega = (0.74734 - 0.17792i)/2M \), with \( M \) the hole’s mass. From the perturbation equations, we have deduced the mode’s horizon vorticity: \( \mathcal{B}_{NN} = \text{Re}[9\sin^2\theta/(2\omega a M^3)] \exp[2\phi - i(\ell + 2M)] \).

[Here \( i \) is the ingoing Eddington-Finklestein time coordinate, and the mode’s Regge-Wheeler radial eigenfunction \( Q(r) \) is normalized to unity near the horizon.] At time
For vectors $\mathbf{E}$ (parallel to solid, positive-tendicity tendex line) and $\mathbf{B}$ (parallel to dashed, positive-vorticity vortex line), $\mathbf{E} \times \mathbf{B}$ is in the wave’s propagation direction.

Now, during and after the merger, the black hole’s near-zone rotating tendex lines (top left inset in Fig. 5) acquire accompanying vortex lines as they travel outward into the wave zone and become gravitational waves, and the rotating near-zone vortex lines acquire accompanying tendex lines. Because of the evolution-equation duality between $\mathbf{E}_ij$ and $\mathbf{B}_ij$, the details of this wave formation are essentially the same for the rotating tendex and vortex lines. Now, in the near zone, the vectors $\mathbf{E}$ and $\mathbf{B}$ along the tendex and vortex lines (Fig. 5) make the same angle with respect to each other as in a gravitational wave (45°) and have $\mathbf{E} \times \mathbf{B}$ in the $-z$ direction. This means that the gravitational waves produced by the rotating near-zone tendex lines and those produced by the rotating near-zone vortex lines will superpose constructively in the $-z$ direction and destructively in the $+z$ direction,
leading to a maximized gravitational-wave momentum flow in the $-z$ direction and maximized black-hole kick in the $+z$ direction. An extension of this reasoning shows that the black-hole kick velocity is sinusoidal in twice the angle between the merged hole’s near-zone rotating vortexes and tendexes, in accord with simulations.

**Conclusions.**—In our BBH simulations, the nonlinear dynamics of curved spacetime appears to be dominated by (i) the transfer of spin-induced frame-drag vortexes from the initial holes to the final merged hole, (ii) the creation of two large vortexes on the merged hole associated with the orbital angular momentum, (iii) the subsequent sloshing, diffusion, and/or rotational motion of the spin-induced vortexes, (iv) the formation of strong negative $E_{\text{NN}}$ poloidal tendexes on the merged horizon at the locations of the original two holes, associated with the horizon’s elongation, and a positive $E_{\text{NN}}$ tendex at the neck where the merger occurs, and (v) the oscillation, diffusion, and/or circulatory motion of these tendexes.

We conjecture that there is no other important dynamics in the merger and ringdown of BBHs. If so, there are important consequences: (i) This could account for the surprising simplicity of the BBH gravitational waveforms predicted by simulations. (ii) A systematic study of frame-drag vortexes and tidal tendexes in BBH simulations may produce improved understanding of BBHs, including their waveforms and kicks. The new waveform insights may lead to improved functional forms for waveforms that are tuned via simulations to serve as templates in LIGO-VIRGO data analysis. (iii) Approximation techniques that aim to smoothly cover the full spacetime of BBH mergers (e.g., the combined post-Newtonian and black-hole-perturbation theory method [17]) might be made to capture accurately the structure and dynamics of frame-drag vortexes and tidal tendexes. If so, these approximations may become powerful and accurate tools for generating BBH waveforms.

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5. [http://www.black-holes.org/headon05aa.html](http://www.black-holes.org/headon05aa.html).