Astrophysics of Black Holes

Igor D. Novikov‡
Institute of Applied Mathematics, Academy of Sciences, Moscow

Kip S. Thorne‡
California Institute of Technology, Pasadena

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Contents

1 Introductory Remarks .................................................. 347
  2.1 Thermal Bremsstrahlung ("Free-Free Radiation" From a Plasma) ..... 347
  2.2 Free-Bound Radiation ............................................. 355
  2.3 Thermal Cyclotron and Synchrotron Radiation ..................... 357
  2.4 Electron Scattering of Radiation .................................. 360
  2.5 Hydrodynamics and Thermodynamics .............................. 362
  2.6 Radiative Transfer ................................................ 370
  2.7 Shock Waves ...................................................... 379
  2.8 Turbulence ....................................................... 383
  2.9 Reconnection of Magnetic Field Lines ............................ 384

3 The Origin of Stellar Black Holes .................................... 386

4 Black Holes in The Interstellar Medium .............................. 389
  4.1 Accretion of Noninteracting Particles onto a Nonmoving Black Hole ...................................................... 389
  4.2 Adiabatic, Hydrodynamic Accretion onto a Nonmoving Black Hole ...................................................... 390
  4.3 Thermal Bremsstrahlung from the Accreting Gas .................... 396
  4.4 Influence of Magnetic Fields and Synchrotron Radiation .......... 397
  4.5 Interaction of Outflowing Radiation with the Gas ................ 399
  4.6 Validity of the Hydrodynamical Approximation .................... 401
  4.7 Gas Flow Near the Horizon ....................................... 402
  4.8 Accretion onto a Moving Hole .................................... 404
  4.9 Optical Appearance of Hole: Summary ............................ 407

5 Black Holes in Binary Star Systems and in the Nuclei of Galaxies .................. 408
  5.1 Introduction ...................................................... 408
  5.2 Accretion in Binary Systems: The General Picture ............... 409
  5.3 Accretion in Galactic Nuclei: The General Picture ............... 420
  5.4 Properties of the Kerr Metric Relevant to Accreting Disks .......... 422
  5.5 Relativistic Model for Disk: Underlying Assumptions ............ 424
  5.6 Equations of Radial Structure .................................... 427
  5.7 Equations of Vertical Structure .................................. 431
  5.8 Approximate Version of Vertical Structure ....................... 434
  5.9 Explicit Models for Disk ........................................ 435
  5.10 Spectrum of Radiation from Disk ................................ 438
  5.11 Heating of the Outer Region by X-Rays from the Inner Region .... 443
  5.12 Fluctuations on the Steady State Model ........................ 443
  5.13 Supercritical Accretion ...................................... 444
  5.14 Comparison with Observations ................................... 445

345
1 Introductory Remarks

To analyze astrophysical aspects of black holes one must bring input from two directions: from the elegant realm of general relativity (lectures of Hawking, Carter, Bardeen, and Ruffini), and from the more mundane realm of astrophysics and plasma physics (§2 of these lectures). We shall here combine these inputs to discuss the origin of stellar black holes (§3); and to analyze observable aspects of black holes in the interstellar medium (§4), in binary star systems and the nuclei of galaxies (§5), and in cosmological contexts (§6).

Readers with strong astrophysical backgrounds will wish to skip over our brief treatment of the fundamentals of astrophysics and plasma physics (§2), and dig immediately into our discussion of black-hole astrophysics (§§3-8). However, for the reader whose previous training has focussed primarily on gravitation theory, the material of §2 is an important prerequisite for understanding the rest of the notes.

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2.1 Thermal Bremsstrahlung (“Free-Free Radiation” From a Plasma)

Basic references chapter 6 of Shkarofsky, Johnston, and Bachynski (1966); chapter 15 of Jackson (1962); §4 of Ginzburg (1967).

Notation for fundamental constants

\begin{align*}
  h & \quad \text{Planck's constant} \\
  m_e & \quad \text{rest mass of electron} \\
  m_p & \quad \text{rest mass of proton} \\
  e & \quad \text{charge of electron} \\
  c & \quad \text{speed of light} \\
  \alpha & \quad \text{fine-structure constant} \\
  r_e & \quad \text{classical electron radius} \\
  R_y & \quad \text{Rydberg energy} \\
  Z & \quad \text{charge of ions in plasma} \\
  A & \quad \text{atomic weight of ions} \\
  k & \quad \text{Boltzmann's constant} \\
  \gamma & \quad \text{Euler constant} (= 1.78 \ldots)
\end{align*}

Radiation from a single classical collision

Consider a single ion of charge \( Z \), at rest in the laboratory frame. Let a single electron of speed \( v \ll c \) (nonrelativistic) scatter off the nucleus (Coulomb scattering), with impact parameter \( b \). Let \( J(\omega) \) be the total energy per unit circular frequency, \( \omega \), radiated by the electron as it scatters. A simple classical
calculation [e.g. Jackson (1962), p. 507] gives
\[ I(\omega) = \frac{2}{3\pi} \frac{e^2}{c^3} |\Delta v_\omega|^2, \]
(2.1.1)
where $\Delta v_\omega$ is the change in electron velocity during a time $\tau = 1/\omega$ centered about the electron’s point of closest approach to the nucleus.

It is useful to examine two limiting cases: small-angle scattering ($\theta \ll 1$) corresponding to large impact parameter ($b \gg b_x-1$); and large-angle scattering ($\theta \sim 2\pi$), corresponding to small impact parameter ($b \ll b_x-1$). The impact parameter, $b_x-1$, which separates small-angle from large-angle scattering, is given by
\[ Z\epsilon^2/b_{x-1} = \frac{1}{2} m_e v_0^2. \]
(2.1.2)

For small-angle scattering the total scattering angle $\theta$ is
\[ \theta = \left( \frac{\Delta v}{v} \right) \frac{Z\epsilon^2}{\frac{1}{2} m_e v^2} = \frac{b_{x-1}}{b}, \]
(2.1.3)
and the time during which the scattering occurs is $\Delta t = b/v$. Hence, for $\tau = 1/\omega \gg \Delta t$, the change in velocity during time $\tau$, $|\Delta v|$, is the full change $|\Delta v|$ of (2.1.3); while for $\tau = 1/\omega \ll \Delta t$ the change is essentially zero. Inserting these changes into equation (2.1.1), we obtain
\[ I(\omega) = 0 \quad \text{for} \quad \omega \gg v/b, \]
\[ I(\omega) = \frac{8}{3\pi} \frac{Z\epsilon^2 c^2}{m_e v^2} \left( \frac{c \theta_0}{vb} \right)^2 \quad \text{for} \quad \omega \ll v/b. \]
(2.1.4)

For large-angle scattering the electron orbit is very nearly a parabola, with “perihelion” at radius
\[ r_p = b^2/b_{x-1} \]
(2.1.5a)
and with speed at perihelion
\[ v_p = c(2Z\epsilon/r_p)^{1/2}. \]
(2.1.5b)
During time intervals $\tau = 1/\omega \ll r_p/v_p$ a negligible amount of velocity change occurs; hence
\[ I(\omega) = 0 \quad \text{for} \quad \omega \gg v_p/r_p = c(2Z\epsilon b_{x-1}^2)^{1/2}/b^3. \]
(2.1.6a)
During time intervals $\tau = 1/\omega \gg r_p/v_p$, but $\tau < b_{x-1}/v$, the electron is initially ($\tau = -\pi/2$) headed directly toward the nucleus with “parabolic” speed
\[ v_{\pi} = \left( \frac{8}{3} \frac{Z\epsilon c^2}{\tau} \right)^{1/3} = 2(4Z\epsilon\omega c^2)^{1/3}; \]
(2.1.6b)
it swings into a sharp turn around the nucleus; and it emerges at $\omega = \pi/2$ with its initial velocity precisely reversed. Consequently, $|\Delta v_\omega|$ is equal to $2v_\omega$, and $I(\omega)$ is
\[ I(\omega) = \frac{32}{3\pi} \frac{e^2}{c^3} \left( \frac{Z\epsilon\omega}{c} \right)^2 \frac{1}{b^2} \quad \text{for} \quad \frac{v}{b_{x-1}} < \omega < \frac{v_p}{b} = \frac{c(2Z\epsilon b_{x-1}^2)^{1/2}}{b^3}. \]
(2.1.6b)

**Radiation from a nonchromatic beam of electrons**

Consider a single ion of charge $Z$, at rest in the laboratory frame. Bombard the ion with a monochromatic beam of electrons with speed $v \ll 1$ and flux (number per unit time per unit area) $\mathcal{S}$. Examine the radiation of frequency $\nu$ (circular frequency $\omega = 2\pi\nu$) emitted by the electrons as they scatter off the nucleus.

Let $\mathcal{P}$ be the total power emitted per unit frequency (ergs sec$^{-1}$ Hz$^{-1}$); and define the emission cross section per unit frequency, $\sigma_{\nu}$, by
\[ \mathcal{P} = (d\sigma/d\nu)\mathcal{S}. \]
(2.1.7)
The dependence of the emission cross section on photon frequency $\nu$ and on electron speed $v$ is most conveniently expressed in terms of the two dimensionless parameters
\[ \frac{\nu}{m_e c^2}, \quad \frac{4\pi \nu^2}{Z^2 R_y} \]
(2.1.8)
On a sheet of paper (Figure 2.1.1) plot $(\nu/m_e c^2)$ vertically, and plot $(4\pi \nu^2/Z^2 R_y)$ horizontally. This 2-dimensional plot can be split into several different regions, in each of which the details of the emission process are different.

**Forbidden region**

The individual photons emitted in each scattering cannot have energies greater than the electron kinetic energy—unless the electron gets captured into a bound state around the ion. But capture produces “free-bound radiation” ($\S$ 2.2) rather than bremsstrahlung. Consequently, the region
\[ (\nu/m_e c^2) > 1 \]
(2.1.9a)
of Figure 2.1.1 is a “forbidden region”; no bremsstrahlung is emitted in this region; $d\sigma/d\nu$ vanishes in this region.

**Photon-discreteness region**

In the region
\[ \frac{1}{2} < (\nu/m_e c^2) < 1 \]
(2.1.9b)
no more than 3 photons can be emitted by each scattering. This “photon discreteness” has a significant effect on the emission cross-section, cutting it down toward zero as one approaches the edge of the boundary of the forbidden region, $(\nu/m_e c^2) = 1$. No classical calculation can reveal such an effect; the
Only large-angle scatterings produce radiation. This region is shown in Figure 2.1.1. The power per unit frequency emitted at a fixed frequency $\nu$ in this region is

$$P_\nu = 2\pi \bar{P}_\omega = 2\pi \int_0^{b_{\text{max}}} (\omega)S2\pi db,$$

(2.11a)

where $I(\omega)$ is given by the large-angle formula (2.1.6), and $b_{\text{max}}$ is the “cutoff” in that formula

$$b_{\text{max}} = \frac{(2\pi^2 Z_0 \bar{b}_0 - \nu^2 \lambda^2)^{1/6}}{\lambda/\nu}.$$

(2.11b)

Performing the integration, dividing by $Shv$ to obtain $d\omega/d\nu$, and reexpressing the result in terms of fundamental atomic constants, one obtains

$$\frac{d\sigma}{dv} = \frac{16\pi^2 Z_0^2 \lambda^2}{3\sqrt{3}} \frac{1}{\nu^3}.$$

(2.12)

A more exact classical calculation gives a slightly different numerical coefficient:

$$\frac{d\sigma}{dv} = \frac{16\pi^2 Z_0^2 \lambda^2}{3\sqrt{3}} \frac{1}{\nu^3}.$$

(2.13)

(Here “$LA$” stands for “large-angle region”.)

In other regions of Figure 2.1.1, one gets other formulas for $d\sigma/d\nu$ (see below). It is conventional in all regions to lump the deviations from this large-angle cross section into a correction term $G(\nu, \nu)$ which is called the “Gaunt factor”:

$$G(\nu, \nu) = \frac{d\sigma}{dv} - \frac{d\sigma}{dv}_{LA}.$$

(2.14)

Thus, in the large-angle region $G = 1$; and in the forbidden region $G = 0$. It turns out that throughout the photon-discreteness portion of the large-angle region, $G$ remains approximately 1; see Figure 2.1.1.

Small-angle, classical region In the region $\nu = 1/\omega \gg b_{\text{max}}/\nu$—i.e.,

$$\frac{1}{\nu} \ll \frac{1}{\omega} \frac{m_0^2}{Z^2 Ry} \frac{1}{\nu} \ll \frac{1}{\omega} \frac{m_0^2}{Z^2 Ry}.$$

(2.14a)
Thus, the uncertainty principle ("\(Z\)" in above equation) prevents the existence of scatterings with \(b < b_{\text{AB}}\) less than the electron de Broglie wavelength

\[
b_{\text{AB}} = \hbar/m_e v.
\]  
(2.1.20)

If \(b_{\text{AB}} < b_{\text{AB}}\), this limitation has no effect on small-angle scatterings; and the small-angle, classical Gaunt factor (2.1.19) is valid. If \(b_{\text{AB}} > b_{\text{AB}}\), the uncertainty principle comes into play, and one must use \(b_{\text{AB}}\) rather than \(b_{\text{AB}}\) as the lower limit on the small-angle integral (2.1.17) for \(d\sigma/dv\). The result is

\[
\frac{d\sigma}{dv} = \frac{16 \alpha^2 e^2 E^2 Z^2}{3 v^2} \ln \left( \frac{b_{\text{max}}}{b_{\text{AB}}} \right).
\]  

(2.1.21)

A more exact, quantum mechanical calculation gives a slightly different argument in the logarithm,

\[
\frac{d\sigma}{dv} = \frac{16 \alpha^2 e^2 E^2 Z^2}{3 v^2} \ln \left( \frac{2b_{\text{max}}}{b_{\text{AB}}} \right),
\]  

(2.1.21a)

corresponding to the Gaunt factor

\[
G(\nu, v) = \frac{\sqrt{3}}{\pi} \ln \left( \frac{2b_{\text{max}}}{b_{\text{AB}}} \right) = \frac{\sqrt{3}}{\pi} \ln \left( \frac{4m_e^2}{\hbar^2} \right).
\]  

(2.1.22)

The "small-angle, uncertainty-principle region", in which this expression holds, is separated from the "small-angle, classical region", where (2.1.19) is valid, by the line

\[
b_{\text{AB}}/b_{\text{AB}} = (\frac{1}{2} m_e^2 Z^2) / (Z^2 R_Y) = 1
\]  

(2.1.23)

See Figure 2.1.1.

From Figure 2.1.1 it is clear that the uncertainty principle cannot have any effect on the large-angle region.

In the "photon-discreteness" portion of the small-angle, uncertainty-principle region, discreteness effects modify the Gaunt factor (2.1.22) into the form

\[
G(\nu, v) = \frac{\sqrt{3}}{\pi} \ln \left( \frac{\nu + \nu Y}{\nu - \nu Y} \right) = \frac{\sqrt{3}}{\pi} \ln \left[ \frac{4m_e^2 (\nu + \nu Y)^2}{\hbar^2} \right].
\]  

(2.1.24a)

Here \(\nu Y\) and \(\nu Y\) are the electron speeds before and after the collision:

\[
\nu_Y = \nu, \quad \frac{1}{2} m_e \nu_Y^2 = \frac{1}{2} m_e \nu^2 - \hbar v.
\]  

(2.1.24b)

[Expression (2.1.24) is the result of a quantum-mechanical Born-approximation calculation, valid throughout the small-angle, uncertainty-principle region.]

Radiation from a plasma

Consider a plasma which may contain a variety of ionic species. Focus attention on radiation from all ions of a particular type, with charge \(Z_e\) and atomic weight \(A\). Let \(C_i\) be the concentration by mass of the species of atoms which gives rise to this ion, and let \(f_i\) be the fraction of all such atoms ionized to the state of interest. Let \(f_i\) be the number of unbound electrons per baryon in the gas. Then

\[
\eta_i = (\text{number of ions per unit volume}) = f_i C_i \rho_0 (A/m_p).
\]  

(2.1.25)

\[
\eta = (\text{number of electrons per unit volume}) = f_i \rho_0 (m_e/m_p).
\]  

Here \(\rho_0\) is the density of rest mass. (Of course, \(f_i \ll 1; f_i \ll 1; C_i \ll 1\).) The ions and electrons both have Maxwell velocity distributions, but because \(m_e \ll m_p\), the velocities of the electrons and their accelerations during Coulomb scattering are far greater than those of the ions. Therefore, in calculating bremsstrahlung one can regard the ions as at rest. The number of ions per unit mass of plasma is \(f_i/m_p\), so the total power per unit frequency emitted from one gram of plasma by electron-ion scatterings is

\[
E_v = \left( \sum_{i} \frac{\text{number of ions per unit mass}}{\text{number of electrons that have speeds } v \text{ in range } dv} \times \text{number density of electrons} \times \frac{d\sigma}{dv} \right) dv
\]  

(2.1.26)

Here \(T\) is the temperature in the plasma. By inserting into the integrand expressions (2.1.12) and (2.1.13) for \(d\sigma/dv\) and performing the integration, one obtains

\[
E_v = \frac{32\pi e^2}{3} \left( \frac{C_{\text{eff}} e^2 Z^2}{A} \right) \frac{m_0^2}{m_e^2} \rho_0 \frac{2m_e^2}{3kT} \frac{Z^2}{3kT} e^{-h\nu/kT} G(\nu, T)
\]  

(2.1.27)

\[
= \left( 2.5 \times 10^{10} \, \text{erg sec}^{-1} \right) \left( \frac{C_{\text{eff}} e^2 Z^2}{A} \right) \frac{m_0^2}{m_e^2} \rho_0 \frac{2m_e^2}{3kT} \frac{Z^2}{3kT} e^{-h\nu/kT} G(\nu, T).
\]  

(2.1.27a)
Here $T_K$ is temperature measured in "K, and $ar{G}(v, T)$ is the "Maxwell-Boltzmann-averaged Gaunt factor", also called the "mean Gaunt factor":

$$
\bar{G}(v, T) = \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} G \left( v, \frac{3m_e^2}{2m_e^2 + m_e^2} \right)^{1/2} e^{-\frac{1}{2} v^2} dv.
$$

(2.1.28)

Corresponding to the 2-dimensional plot of Gaunt factors (Figure 2.1.1) one can make a 2-dimensional plot of mean Gaunt factors (Figure 2.1.2). It is straightforward to calculate the mean Gaunt factors shown there (up to quantities of order unity) from the Gaunt factors of Figure 2.1.1. Notice that,

$$
\bar{G}(v, T) \approx G(v = x kT/h, T) e^{-x} dx.
$$

(2.1.30)

From a value of $\bar{G} \approx 1.2$ at $kT/Z^2 Ry \sim 0.01$, it increases gradually to a maximum of 1.42 at $kT/Z^2 Ry \sim 1$, and then decreases gradually to an asymptotic, high-temperature value of 1.103 at $kT/Z^2 Ry > 100$; see Figure 2 of Green (1959).

In a plasma at nonrelativistic temperatures, the radiation from electron-electron collisions and from ion-ion collisions is negligible compared to electron-ion radiation. This is because (i) if identical particles scatter, the first time derivative of the electric dipole moment is conserved, so only radiation of quadrupole order and higher [with intensity $\sim (v/c)^2$ less than dipole radiation] can be emitted; (ii) the speeds and accelerations of colliding ions are far less than those of electrons because of their much larger masses.

Equations (2.1.27) and (2.1.29), together with Figures 2.1.2 and 2.1.3, are the chief results of astrophysical interest from this section. When dealing with atoms that are only partially ionized, one must sometimes correct these results for screening of the nuclear charge by the bound electron cloud; see, e.g., Ginzburg (1967). For temperatures approaching and exceeding $kT = m_e v^2$, relativistic effects and electron-electron collisions modify the emissivity (2.1.29) to the form

$$
e = \left( \frac{5.2 \times 10^{30} \text{ergs}}{\text{g sec}} \right) \left( \frac{C_a f a_0 Z^2}{A} \right) \left( \frac{\rho_0}{\text{g/cm}^3} \right) T_k^{1/2} \left( 1 + 4.4 \times 10^{-10} T_k \bar{G}(T) \right).
$$

(Ginzburg 1967).

2.2 Free-Bound Radiation

Basic references  §4-17 of Aller (1963); Brussard and van de Hulst (1962).

Basic physics and formulas

When an electron of kinetic energy $\frac{1}{2} m_e v^2 \leq Z^2 R_y$...
impinges on an ion, there is a significant probability that, instead of scattering with the emission of bremsstrahlung, it will get captured into a bound state with an accompanying emission of "free-bound" radiation.

We shall not compute here the details of the radiation. Instead we cite the results of such a computation in the above references.

Consider a plasma at temperature $T$ and density of rest mass $\rho_0$. Let $C_A$ be the concentration of mass of a particular atom in the plasma; let $f_s$ be the fraction of all such atoms ionized into a given state, with charge $Ze$; and let $f_e$ be the number of free electrons per baryon [cf. eq. (2.1.25)]. Consider the radiation emitted by the free electrons into a particular bound state about the ion; and let $n_q$ be the principal quantum number of that state, and $E_i$ its ionization energy. Then the emissivity due to such transitions is

$$\varepsilon_{\nu} = e^2 \left( \frac{C_A f \Sigma^\infty}{An_q^2} \right) \frac{\alpha^2 m_e^2 c^2}{m_p^2} \frac{\rho_0}{3kT} \frac{2m_e c^2}{3kT} \frac{e^{-\nu h - E_i kT} G_M}{2}$$

$$= \left( 3.9 \times 10^{-3} \text{ ergs} \text{ g sec Hz}^{-1} \right) \left( \frac{C_A f \Sigma^\infty}{An_q^2} \right) \left( \frac{\rho_0}{gm^{-3}} \right) \frac{e^{-\nu h - E_i kT} G_M}{2}$$

Here $G_M$ is a "bound-free Gaunt factor", analogous to the "free-free" Gaunt factors of §2.1. It depends on the kinetic energy

$$\frac{1}{2} m_e c^2 = h\nu - E_i$$

of the electron that is captured, and on the structure of the bound state (which one characterizes by its quantum numbers $n_q, l_q, \ldots$):

$$G_M = G_M(h\nu - E_i; n_q, l_q, \ldots)$$

Gaunt factors for various bound-free transitions of astrophysical interest are tabulated and graphed by Karzas and Latter (1961). Energy conservation requires that all photons emitted have energies greater than $E_i$; consequently,

$$G_M = 0 \text{ for } h\nu - E_i < 0$$

In general, $G_M$ "turns on" discontinuously at $h\nu = E_i$, with an initial value between $\sim 0.5$ and $\sim 1.5$:

$$G_M \approx 1 \text{ for } 0 < h\nu - E_i < E_i$$

For $h\nu - E_i \lesssim 2Z^2\text{Ry}$, $G_M$ typically remains within an order of magnitude of unity; as $h\nu - E_i$ increases in the region $h\nu - E_i \gtrsim 10 Z^2\text{Ry}$, $G_M$ falls rapidly. See Karzas and Latter (1961).

The total emissivity, $\varepsilon = \int e_{\nu} d\nu$, due to a given free-bound transition is

$$\varepsilon = 8 \sqrt{3} \frac{c^2}{3kT} \left( \frac{C_A f \Sigma^\infty}{An_q^2} \right) \left( \frac{\alpha^2 m_e c^2}{m_p^2} \right) \rho_0 \frac{2m_e c^2}{3kT} \frac{e^{-\nu h - E_i kT} G_M}{2}$$

$$= \left( 4.2 \times 10^{26} \frac{\text{ergs}}{\text{g sec}} \right) \left( \frac{C_A f \Sigma^\infty}{An_q^2} \right) \left( \frac{\rho_0}{gm^{-3}} \right) T_k^{-1/2} G_M$$

where $G_M$, the mean Gaunt factor, depends only on temperature and on the bound state, and is approximately equal to one

$$G_M \approx \int_0 G_M(h\nu - E_i = nKT; n_q, l_q, \ldots) e^{-\nu h} d\nu \approx 1$$

(2.2.7)

The total emissivity (2.2.6) due to a given free-bound transition, compared to the total emissivity (2.1.29) of free-free radiation from the same type of ion, is

$$\frac{\varepsilon_{\text{fb}}}{\varepsilon_{\text{ff}}} \approx \frac{e^2}{\gamma} \left( \frac{8 \times 10^8}{T} \right)$$

Ginzburg (1967) states that for astrophysical plasmas the ratio of total free-bound radiation to total free-free radiation (summed over all states and all ions) generally is less than

$$\frac{\varepsilon_{\text{fb total}}}{\varepsilon_{\text{ff total}}} < \frac{8 \times 10^8}{T}$$

(2.2.9)

2.3 Thermal Cyclotron and Synchrotron Radiation

Basic references Jackson (1962); Ginzburg (1967).

Power radiated by a single electron

Consider an electron of speed $v$ and total mass-energy $\gamma m_e c^2$.

$$\gamma = (1 - v^2)^{-1/2}$$

(2.3.1)

spiralling in a magnetic field of strength $B$. The Lorentz 4-acceleration of the electron is

$$\alpha^0 = 0, \quad \alpha = (e/m_e) \gamma v \times B$$

(2.3.2)

Consequently, the total power radiated—as obtained from the standard Lorentz-invariant equation

$$\frac{dE}{dt} = \frac{2e^2}{3c^3 \alpha^2}$$

(2.3.3)
\[ \frac{dE}{dt} = \frac{2}{3} \frac{r^2}{c} (\gamma u_\perp)^2 B^2. \]  
\[ (2.3.4) \]

Here \( u_\perp \) is the component of the electron velocity perpendicular to the magnetic field. If the electron is nonrelativistic, \( u \ll c \), this radiation is called "cyclotron radiation". If the electron is ultrarelativistic, \( \gamma \gg 1 \), it is called "synchrotron radiation".

**Cyclotron radiation from a nonrelativistic plasma.**

Consider a plasma with temperature \( kT < m_e c^2 \), i.e. \( T < 6 \times 10^6 \text{K} \).

and with a magnetic field of strength \( B \). The radiation from protons spiralling in the magnetic field is smaller by a factor \((m_p/m_e)^2 \approx 10^{-10}\) than that from electrons, since

\[ \frac{dE}{dt} \propto \frac{v^2}{m_e} \propto \frac{1}{m_e} \]  
\[ (2.3.6) \]

(Recall: in thermal equilibrium the mean kinetic energies of protons and electrons are the same.) Hence, we can ignore protons and other ions when calculating cyclotron radiation. Let \( f_p \) be the number of unbound electrons per baryon in the plasma. Then the total emissivity (ergs per second per gram of plasma) is

\[ e = \frac{f_p}{m_p} \frac{dE}{dt}, \]  
\[ (2.3.7) \]

where \( \langle \cdot \rangle \) denotes an average over the Maxwell-Boltzmann velocity distribution of the electrons. Since \( \gamma = 1 \), since 2 of the 3 spatial directions are orthogonal to \( B \), and since the mean kinetic energy per proton is \( 2kT \), we have

\[ \langle (\gamma u_\perp)^2 \rangle = \langle v^2 \rangle = 2kT/m_e. \]  

\[ (2.3.8) \]

Combining this with equations (2.3.4) and (2.3.7), we obtain

\[ e = \frac{4}{3} \left( \frac{kT}{m_p c^2} \right) \left( \frac{kT}{m_e c^2} \right) B^2 \]  
\[ (2.3.9) \]

where \( kT \) is temperature in K and \( B_G \) is magnetic field in Gauss.

Each electron emits its radiation monochromatically, with the cyclotron frequency (orbital frequency of electron’s spiral motion)

\[ \nu_{\text{cyc}} = \frac{eB}{2\pi m_e c} = (2.79 \text{ MHz}) T_k B_G. \]  
\[ (2.3.10) \]

Thus, if the magnetic field is uniform, the radiation will be monochromatic with this frequency. But in Nature the magnetic field will always be inhomogeneous, and the spectrum will show a spread proportional to the spread in \( B \).

(Of course, there are always other sources of spread in the spectrum—e.g., doppler shifts and relativistic generation of harmonics.)

**Synchrotron radiation from a relativistic plasma**

Consider a plasma with temperature \( kT > m_e c^2 \), i.e. \( T > 6 \times 10^6 \text{K} \),

and with a magnetic field of strength \( B \). As in the nonrelativistic case, radiation from protons and ions is totally negligible. The total emissivity due to electrons and positrons (recall: at \( kT > m_e c^2 \) there will be many electron-positron pair")

is given, as before, by equation (2.3.7); but now the factor \( f_p \) must be the number of electrons and positrons per baryon, and the relevant ultrarelativistic Maxwell-Boltzmann average is

\[ \langle (\gamma u_\perp)^2 \rangle = \langle v^2 \rangle = \frac{1}{12} \left( \frac{kT}{m_e c^2} \right)^2 = \left( \frac{kT}{m_e c^2} \right)^2 \]  
\[ (2.3.12) \]

[One evaluates \( \langle \gamma^2 \rangle \) using the distribution function in phase space

\[ \mathcal{N} = \frac{dN}{d^2x d^2p} \propto e^{-E/kT} = e^{-m_e c^2/kT}, \]  

with

\[ d^3p = 4\pi c^{-3} E^2 dE \]  

in ultrarelativistic limit,

in particular

\[ \langle \gamma^2 \rangle = \left( \left\langle \frac{E}{m_e c^2} \right\rangle \right)^2 \propto \frac{1}{(m_e c^2)^2} \left[ \int E^2 e^{-E/kT} E^2 dE \right] = 12 \left( \frac{kT}{m_e c^2} \right)^2. \]  

Combining equations (2.3.12), (2.3.7), and (2.3.4), we obtain for the total emissivity of the plasma

\[ e = \frac{16}{3} \left( \frac{m_p}{m_e} \right) \left( \frac{kT}{m_e c^2} \right)^2 B^2 \]  
\[ (2.3.13) \]

\[ = (2.2 \times 10^{-10} \text{ ergs/g sec}) f_p T_k B_G^2. \]

In the ultrarelativistic case the electrons do not emit monochromatically. An electron of energy \( m_m c^2 \) beams most of its radiation into a forward cone of half-angle \( \alpha = 1/\gamma \) (special-relativistic "headlight effect"). Since the electron is headed toward the observer with speed \( v \) when its cone is directed toward him,

\[ \dagger \text{ The case of an optically thin plasma is of particular interest in astrophysics. The kinetic theory of the creation and annihilation of positrons in this case is somewhat complex; see, e.g., Binnovozsky-Kogan, Zel’dovich, and Sunyaev (1971).} \]
the cone sweeps past the observer in time
\[ \Delta t = (1 - \nu^2)^{-\frac{2\alpha}{\omega_{\text{circ}}}} \left(1 - \nu^2\right)^{\frac{1}{3}} \frac{1}{\omega_{\text{circ}}} \cdot \gamma^3 \omega_{\text{circ}}. \tag{2.3.14} \]

Here \( \omega_{\text{circ}} \) is the angular velocity of the electron in its spiraling orbit
\[ \omega_{\text{circ}} = \frac{eB}{\gamma m_e c}. \tag{2.3.15} \]

Thus, the radiation comes in short bursts, of duration \( \sim \Delta t = 1/(\gamma^3 \omega_{\text{circ}}) \), separated by long intervals of time \( 2\nu^2/\omega_{\text{circ}} \). When Fourier analyzed, such radiation must be concentrated near the "critical frequency".
\[ \nu_{\text{crit}} = \gamma^3 \omega_{\text{crit}} = \gamma^3 \frac{eB}{m_e c}. \tag{2.3.16} \]

A detailed calculation [chapter 14 of Jackson (1962)] reveals a fairly broad spectrum which rises as \( \nu^{1/2} \) at low frequencies, which peaks at
\[ \nu_{\text{peak}} = 0.29 \nu_{\text{crit}}, \tag{2.3.17} \]

and which decays exponentially, \( \sim \nu^{1/2} e^{-2\nu/\nu_{\text{crit}}} \), at \( \nu \gg \nu_{\text{crit}} \).

When averaged over the relativistic plasma, such a spectrum will have a broad peak, with maximum located at
\[ \nu_{\text{peak, plasma}} \approx \frac{12(kT/m_e c^2)^2}{(kT/m_e c^2)^2 + 100 \text{ MHz}}. \tag{2.3.18} \]

Here \( T_10 \) is temperature in units of \( 10^{10} \text{ K} \). At \( \nu \gg \nu_{\text{peak, plasma}} \), the spectrum will decay exponentially, and if \( \nu \ll \nu_{\text{peak, plasma}} \), it will rise as a power law.

2.4 Electron Scattering of Radiation

Basic references §14.7 of Jackson (1962b); §12.3 of Leighton (1959); Kompaneets (1957); Weyman (1965); Sunyaev and Zel’dovich (1973).

Basic physics and formulas

In discussing electron scattering, we shall confine attention to the nonrelativistic case—i.e. we shall demand that the temperature of the gas and the photon frequencies satisfy
\[ kT \ll m_e c^2 (T \ll 6 \times 10^8 \text{ K}) \quad \text{and} \quad \nu \ll m_e c^2 = 500 \text{ keV}. \tag{2.4.1} \]

The differential cross section for a free electron to scatter a photon has the form
\[ \frac{d\sigma_{ee}}{d\Omega} = \frac{1}{4} \alpha \left(1 + \cos^2 \theta\right), \tag{2.4.2} \]

where \( \theta \) is the angle between incoming photon and outgoing photon. The total cross section ("Thompson cross section"), obtained by integrating over all directions, is
\[ \sigma_{ee} = \frac{(8\pi/3)\alpha^3}{\gamma^2} = 0.665 \times 10^{-24} \text{ cm}^2. \tag{2.4.3} \]

It is important for astrophysical applications that the scattering cross section is "color blind" (no dependence on frequency). See, e.g., §4.

It is also important that the differential cross section is symmetric between forward and backward directions. This guarantee, for example, that on the average a scattered photon transmits all of its momentum to the electron.
\[ \langle \Delta p \rangle = p_y = (\hbar/2)e_n. \tag{2.4.4a} \]

(Here \( e_n \) is a unit vector in the direction of the initial photon motion.) By contrast, only a tiny fraction of the photon energy is transmitted to the electron. In a frame where the electron is initially at rest, it receives a kinetic energy
\[ \Delta E_e = (\hbar/2m_e c^2) - \hbar(1 - \cos \theta). \tag{2.4.4b} \]

(Recall: we have assumed \( \hbar/m_e c^2 \ll 1 \).

Consider a monochromatic beam of photons, with frequency \( \nu \), moving through a thermalized plasma of temperature \( T \). Scattering by protons and ions will be negligible compared to scattering by electrons, since
\[ \sigma_{pp} \approx \sigma_{ee} \approx (e^2/m_e c^2)^2 \approx 1/(\text{mass of scatterer})^2. \tag{2.4.5} \]

On the average, how much energy \( \langle \Delta E_e \rangle \) is transmitted to the electron in a single scattering? The answer for electrons nearly at rest \( (kT \ll \hbar \nu) \) is obtained by averaging expression (2.4.4b) over all angles. Because of the forward-backward symmetry of the differential cross section, cost \( \theta \) averages to zero and one obtains
\[ \langle \Delta E_e \rangle = (\hbar/2m_e c^2) \hbar \nu \cdot kT \ll \hbar \nu. \tag{2.4.6} \]

One can calculate \( \langle \Delta E_e \rangle \) for higher temperatures by invoking conservation of 4-momentum, and averaging over all angles and electron speeds. However, such a calculation is rather long and messy. To obtain the answer more easily, one can use the following trick: An examination of the law of 4-momentum conservation convinces one that \( \langle \Delta E_e \rangle \) must have the temperature dependence
\[ \langle \Delta E_e \rangle \approx (\hbar/2m_e c^2) (kT - akT), \tag{2.4.7} \]

where \( a \) is a constant to be calculated. (Thus, if \( \hbar > a kT \), the electrons get heated by the photons; if \( \hbar < a kT \), they get cooled.) This law will reduce to expression (2.4.6) for \( kT \ll \hbar \nu \) if and only if the proportionality constant is \( \hbar/2m_e c^2 \).

To calculate the constant \( a \), imagine the following experiment: Place a large number of photons and electrons into a box with perfectly reflective walls.
Require that the photons and electrons interact only by electron scattering. Then photons cannot be created or destroyed, only scattered. Hence, when thermal equilibrium is reached, the photons acquire a Boltzmann energy distribution, rather than a Planck distribution. (Recall that stimulated emissions are responsible for Planckian deviations from the Boltzmann law; §2.5). Since photons have zero rest mass, their Boltzmann distribution

$$N = \frac{dN}{d^3x \, d^3p} = \frac{\alpha}{e^{(\beta/kT)} - 1} \times \frac{d^3p}{4\pi e^{-\beta/kT}}$$

corresponds to a number of photons per unit frequency given by

$$\frac{dN}{dp} = \frac{\alpha}{e^{(\beta/kT)}}$$  \hspace{1cm} (2.4.8)

and corresponds to

$$\langle \nu \rangle = \frac{1}{\alpha}$$

Thus, for our thought experiment the energy transfer in each collision, expression (2.4.7) averaged over the equilibrium photon distribution, is

$$\langle \Delta E \rangle = \frac{3kT}{\alpha}$$

But in equilibrium there must be no average energy transfer between photons and electrons; \(\Delta E\) must be zero. This condition tells us that \(\alpha = 4\).

Thus, turning back to the general situation, we conclude that monochromatic photons passing through a plasma of temperature \(T\) transfer an average energy per collision

$$\langle \Delta E \rangle = \frac{(h\nu / m_e c^2) (h\nu - 4kT)}{4\pi e^{(\beta/kT)}}$$  \hspace{1cm} (2.4.9)

to the electrons. When \(4kT > h\nu\), the photon energies get boosted by the collisions, and one says that the radiation is being "Comptonized"; see §5.10;

This concludes, for the moment, our discussion of the interaction between radiation and plasmas. We must point out that we have ignored a number of plasma effects and instabilities which can be important in astrophysical situations. See, e.g., Bekeff (1966), and Kaplan and Setovich (1972).

2.5 Hydrodynamics and Thermodynamics

Basic references §22.2 and 22.3 of Minser, Thorne, and Wheeler (1973); Ellis (1971); Lichnerowitz (1967) and references cited therein. We adopt the notational conventions of Minser, Thorne, and Wheeler (cited henceforth as MTW), including signature "+---"; geometrized units (\(c = G = 1\)); Greek indices ranging from 0 to 3 and Latin from 1 to 3; "hats" on indices that refer

to local orthonormal frames, e.g. \(u_\rho^\gamma\) and \(T^{\rho\gamma}\); extra-bold, sans-serif type for 4-vectors and 4-tensors, e.g. \(u\) and \(T\); normal bold-face type for 3-vectors and 3-tensors (local Euclidean geometry).

Parameters describing the "fluid"

"LRF" local rest frame of the baryons; i.e. local orthonormal frame in which there is no net baryon flux in any direction.

\(u\) 4-velocity of LRF; i.e. 4-velocity of the "fluid".

\(n\) number density of baryons (number of baryons per unit volume) as measured in LRF.

\(m_B\) mean rest mass of a baryon in the fluid; i.e.

$$m_B = \left( \frac{\text{rest mass of hydrogen atom}}{\text{in the form of hydrogen nuclei}} \right) \times \left( \frac{\text{fraction of all baryons that are hydrogen atom}}{\text{are in hydrogen nuclei}} \right)$$

Note: in this section we restrict ourselves to fluids that are chemically homogeneous and in which no nuclear reactions occur ("standard fluid"). Thus, \(m_B\) is constant.

\(\rho_0\) rest-mass density, defined by

$$\rho_0 \equiv m_B n$$  \hspace{1cm} (2.5.2)

\(V\) specific volume, i.e. volume per baryon, defined by

$$V \equiv 1/n$$  \hspace{1cm} (2.5.3)

\(V_0\) specific volume, i.e. volume per unit rest mass, defined by

$$V_0 \equiv 1/\rho_0$$  \hspace{1cm} (2.5.4)

\(\rho\) total density of mass-energy, as measured in LRF.

\(\Pi\) specific internal energy, defined by

$$\rho = \rho_0 (1 + \Pi)$$  \hspace{1cm} (2.5.5)

Note: here and throughout this section we set the speed of light equal to one.

\(p\) isotropic pressure, as measured in LRF.

\(T\) temperature, as measured in LRF.

\(s\) entropy per baryon, as measured in LRF.

\(s_0\) entropy per unit mass, as measured in LRF; of course,

$$s_0 = m_B^{-1} s.$$  \hspace{1cm} (2.5.6)
\[ \mu = \text{chemical potential, as measured in LRF; defined by} \]
\[ \mu = \left( \frac{\delta p}{\delta n} \right)_s = \frac{\rho + p}{n}. \]  
(2.5.7)

(The second equality follows from the first law of thermodynamics, below.)

\[ q = \text{flux of energy (due to heat conduction, radiation, convection, etc.) as measured in LRF. This flux (ergs cm}^{-2} \text{sec}^{-1}) \text{is a purely spatial vector as measured in LRF; i.e.} \]
\[ q \cdot u = 0 \]  
(2.5.8)

\[ S = \text{entropy density-flux vector. In LRF this vector has time component equal to} \]
\[ S_s = ns, \text{and space components equal to the entropy flux,} \]
\[ S_i = q_i/T. \text{Hence, in frame-independent notation} \]
\[ S = ru + q/T. \]  
(2.5.9)

Note that
\[ \nabla \cdot S = \left( \frac{\text{rate at which entropy is being generated}}{\text{per unit volume as measured in LRF}} \right) \]  
(2.5.10)

**First law of thermodynamics**

Follow a fluid element, containing \( A \) baryons, along its world tube. The total mass-energy in the fluid element, \( pA/n \), changes as a result of compression (change in volume, \( A/n \)) and as a result of influx of heat:
\[ d(pA/n) = -pdA/A + TdA/\partial t. \]  
(2.5.11)

[Here and throughout this section we assume for simplicity no internal generation of entropy in the fluid element—e.g., no irreversible chemical reactions; this enables us to write the influx of heat in terms of the change in entropy, \( TdA/\partial t). \]

One often uses \( \mu = const. \) and \( \mu = (\rho + p)/n \) to rewrite this first law of thermodynamics in the equivalent forms
\[ dp = \frac{\rho + p}{n} \text{dn} + nTds, \]  
(2.5.12)

\[ du = Vdp + Tds. \]  
(2.5.12')

From the first law one can read off partial derivatives, e.g.
\[ \left( \frac{\partial p}{\partial n} \right)_s = \frac{\rho + p}{n}, \quad \left( \frac{\partial u}{\partial \rho} \right)_p = T. \]

**Fundamental relation and equations of state**

The peculiar thermodynamic properties of the particular fluid being studied are determined by a “fundamental thermodynamic relation”
\[ \rho = \rho(n, s) \quad \text{or} \quad \mu = \mu(p, s). \]  
(2.5.13)

Once this relation has been specified—and once the constant \( m_B \) has been specified—one can use the first law of thermodynamics (2.5.12) or (2.5.12') and definitions (2.5.2)–(2.5.7) to derive explicit expressions for all other thermodynamic variables as functions of \( n, s \) or \( p, s \). For example, by combining with the first law one can derive the “equations of state”
\[ T(n, s) = \frac{1}{n} \frac{\partial p}{\partial n}_s, \quad p(n, s) = n \frac{\partial p}{\partial n}_s - \rho. \]

**Adiabatic index and speed of sound**

One defines the adiabatic index \( \Gamma \), by
\[ \Gamma = \left( \frac{\partial p}{\partial n} \right)_s = \frac{\partial p}{\partial V} \frac{1}{\rho} + \frac{\rho + p}{\rho} \frac{\partial \rho}{\partial n} \]  
(2.5.14)

where the third equality follows from the first law of thermodynamics. It turns out that weak adiabatic perturbations (weak “sound waves”) propagate, in the LRF, with ordinary velocity
\[ c_s = \left( \left( \frac{\partial p}{\partial n} \right)_s \right)^{1/2} = \left( \frac{\Gamma - 1}{\Gamma} \right)^{1/2} \]  
(2.5.15)

**Second law of thermodynamics**

Equation (2.5.10) allows one to write the second law of thermodynamics in the form
\[ \nabla \cdot S \geq 0. \]  
(2.5.16)

**Decomposition of 4-velocity**

One decomposes the gradient of the 4-velocity, \( \nabla u \), into its “irreducible” tensorial parts
\[ u_{a\beta} = v_{a\beta} + a_{a\beta} + \frac{1}{2} \partial_{a\beta} - a_{a\beta} \]  
(2.5.17)

Here \( a \) is the 4-acceleration of the fluid
\[ a = \nabla \times u, \quad a_{a\beta} = \frac{\partial u_{\alpha\beta}}{\partial x^a}. \]  
(2.18a)

and \( -\omega_{a\beta} \) is that portion of \( u_{a\beta} \) which is not orthogonal to \( u \). The remainder of \( H_{a\beta} \) (i.e., \( \omega_{a\beta} + a_{a\beta} + \frac{1}{2} \partial_{a\beta} \)) is orthogonal to \( u \); i.e., in the LRF it has only spatial components. This orthogonal part is decomposed into an isotropic
expansion, \( \frac{1}{3} \theta h_{\alpha \beta} \), where \( \theta \) is the "expansion"

\[
\theta \equiv \nabla \cdot \mathbf{u} = u^{\alpha}_{\alpha}
\]  
(2.5.18b)

and \( h_{\alpha \beta} \) is the "projection tensor"

\[
h_{\alpha \beta} \equiv \delta_{\alpha \beta} + u_{\alpha} u_{\beta};
\]  
(2.5.18c)

plus a symmetric, trace-free "shear"

\[
\sigma_{\alpha \beta} = \frac{1}{2} (u_{\alpha, \beta} h_{\gamma \delta} + u_{\beta, \gamma} h_{\delta \alpha}) - \frac{1}{2} g h_{\alpha \beta};
\]  
(2.5.18d)

plus an antisymmetric "rotation" or "vorticity"

\[
\omega_{\alpha \beta} = \frac{1}{2} (u_{\alpha, \beta} h_{\gamma \delta} - u_{\beta, \gamma} h_{\delta \alpha}).
\]  
(2.5.18e)

An observer in the LRF sees all fluid elements in his neighborhood to move with low (nonrelativistic) velocities. Let him use standard Newtonian methods to calculate or measure the expansion \( \theta \), shear \( \sigma \), and rotation \( \omega \) of the fluid; and let a relativist calculate \( \theta, \sigma \), and \( \omega \) in the LRF from the above equations. The two calculations will give the same answers. [See, e.g., of Ellis (1971).]

Frozen-in magnetic field

Consider an ionized plasma which contains a "frozen-in" magnetic field. The field is pure magnetic (no electric field) in the LRF. Therefore it can be described by a magnetic-field 4-vector \( \mathbf{B} \) orthogonal to \( \mathbf{u} \).

\[
\mathbf{B} \cdot \mathbf{u} = 0.
\]  
(2.5.19)

For simplicity we assume that the magnetic permeability of the plasma is the same as that of vacuum; so we do not distinguish between "\( \mathbf{B} \)" and "\( \mathbf{B} \)." For a more general treatment, see Lichnerowicz (1967). As the fluid moves, carrying with it the frozen-in \( \mathbf{B} \), it must change as

\[
\frac{d \mathbf{B}_{\alpha}}{dt} = u_{\alpha} \varepsilon_{\alpha} B_{\beta} + \omega_{\alpha \beta} B_{\beta} + (\sigma_{\alpha \beta} - \frac{1}{3} \theta h_{\alpha \beta}) B_{\beta};
\]  
(2.5.20)

The term \( u_{\alpha} \varepsilon_{\alpha} B_{\beta} \) is required to keep \( B \) orthogonal to \( u \); by the term \( \omega_{\alpha \beta} B_{\beta} \) the rotation of the fluid rotates the field lines; by the term \( (\sigma_{\alpha \beta} - \frac{1}{3} \theta h_{\alpha \beta}) B_{\beta} \) the compression of the fluid orthogonal to \( B \), conserving flux, magnifies \( B \).

Stress-energy tensor

Consider a fluid with isotropic pressure, with shear and bulk viscosity, with energy flowing between fluid elements, with a frozen-in magnetic field. The stress-energy tensor for such a fluid is

\[
T^\alpha_\beta = \rho u^\alpha u_\beta + (p - \frac{1}{2} \theta h_{\alpha \beta}) h_{\alpha \beta} - \eta \sigma_{\alpha \beta} + q_{\alpha \beta} u^\gamma u_\gamma - \frac{1}{8\pi} \left( B^2 u^\alpha u_\beta + B^2 h_{\alpha \beta} - 2B^\alpha B_\beta \right). \]  
(2.5.21)

The term \( \rho u^\alpha u_\beta \) is the total density of mass-energy (excluding only that of the frozen-in \( B \)-field) as measured in the LRF. The term \( p h_{\alpha \beta} \) is the isotropic pressure that would be measured in the LRF if the gas were not changing volume (if \( \theta \) were zero). The quantities \( \xi \) and \( \eta \) are the coefficients of bulk viscosity and of dynamic viscosity, respectively. The term \( -\frac{1}{2} \theta h_{\alpha \beta} \) is the isotropic viscous stress which resists isotropic expansion (\( \theta > 0 \)) or compression (\( \theta < 0 \)) of the fluid. The term \( -2\eta \sigma_{\alpha \beta} \) is the viscous shear stress which resists shearing motions. The term \( -q_{\alpha \beta} u^\gamma u_\gamma \) is the energy flux and momentum flux relative to the LRF. (Note: The energy density and stresses associated with the flowing energy \( q \), as measured in the LRF, are here neglected by comparison with \( \rho u^\alpha u_\beta \) and \( p h_{\alpha \beta} \). This neglect is valid with enormous accuracy in most contexts of interest in these lectures. An exception is the radiation pressure which produces "self-regulation" of accretion onto black holes when the inflowing mass is sufficiently large; see § 5.4 and 5.13.) The term

\[
T^\alpha_\alpha = \frac{1}{8\pi} \left( B^2 u^\alpha u_\beta + B^2 h_{\alpha \beta} - 2B^\alpha B_\beta \right) \]  
(2.5.22)

is the Maxwell stress-energy tensor in the frozen-in \( B \)-field (in the LRF: energy density \( B^2/8\pi \), pressure \( B^2/8\pi \) orthogonal to field lines; tension \( -B^\alpha B_\alpha \) along field lines).

Equations of hydrodynamics

The fundamental equations governing the motion of a fluid in a given gravitational field (spacetime geometry) are (i) the law of baryon conservation

\[
\nabla \cdot (\rho \mathbf{u}) = 0, \text{ i.e. } \frac{d \rho}{dt} = -\rho \nabla \cdot \mathbf{u},
\]  
(i.e., the law of baryon conservation)

\[
\nabla \cdot (\rho \mathbf{u}) = 0, \text{ i.e. } \frac{d \rho}{dt} = -\rho \nabla \cdot \mathbf{u},
\]  
(2.5.23)

The term \( \rho \mathbf{u} \cdot \nabla \mathbf{u} \) is the term of a fluid, carrying with it the frozen-in \( B \), and the term \( \rho \nabla p \) is the density term of the pressure. For a more general treatment, see Lichnerowicz (1967).

(iii) the Euler equations (i.e. law of local momentum conservation)

\[
\mathbf{h} \cdot (\nabla \cdot \mathbf{u}) = 0;
\]  
(2.5.24)

(iv) the laws of thermodynamics, (2.11)-(2.16); (v) the law of evolution for the frozen-in \( B \)-field, (2.20); (vi) the laws of energy transport, which govern \( q \) (see § 2.6 eq. 2.643).

Law of local energy conservation

When one evaluates the law of local energy conservation, \( \mathbf{u} \cdot (\nabla \cdot \mathbf{u}) = 0 \), for the stress-energy tensor (2.5.21), one finds that the Maxwell stress-energy
Euler equation for a perfect fluid

Consider a perfect fluid, flowing adiabatically through spacetime \((\xi = \eta = B = q = 0)\). In this case the Euler equations \(\mathbf{h} \cdot (\nabla \times \mathbf{T}) = 0\) reduce to

\[
(p + \rho) \mathbf{v} = -\mathbf{h} \cdot \nabla \rho.
\]

In words:

\[
\text{inertial mass per unit volume} \times (4\text{-acceleration}) = -\text{pressure gradient projected orthogonal to } \mathbf{u}.
\]

Bernoulli equation

Consider a perfect fluid undergoing stationary, adiabatic flow in a stationary spacetime. More particularly, set \(\xi = \eta = B = q = 0\); assume that spacetime is endowed with a Killing vector field \(\xi\),

\[
\dot{\xi}_0 + \dot{\xi}_\alpha = 0,
\]

which need not be timelike; and assume that the flow is adiabatic, \(dt/d\mathbf{r} = 0\), and stationary in the sense that

\[
L_{\xi} \mathbf{u} = 0, \quad \nabla_{\xi} \mathbf{p} = \nabla_{\xi} \mathbf{p} = \cdots = 0.
\]

Here \(L_{\xi} \mathbf{u}\) is the Lie derivative along \(\xi\). In this case the Euler equations (2.5.30), together with the first law of thermodynamics (2.5.12), imply the relativistic Bernoulli equation

\[
d(\mu u \xi)/dt = 0;
\]

i.e., \(\mu u \cdot \xi\) is constant along flow lines.

Newtonian limit

The Newtonian limit of relativistic hydrodynamics is obtained when, in a nearly global Lorentz frame, the following approximations hold:

\[
\begin{align*}
\mathbf{B} &= (1 + 2\Phi), & |\Phi| &< 1; \\
p/\rho_0 &< 1, & \Pi &< 1, & B^2/\rho_0 &< 1, & v^2 &< 1.
\end{align*}
\]

Here \(\Phi\) is the Newtonian gravitational potential with sign \(\Phi < 0\), and \(v\) is the ordinary velocity of the fluid

\[
v \equiv u_\xi = (u/\rho_0)^{1/2}.
\]

In the Newtonian limit the chemical potential \(\mu\) reduces to

\[
\mu = m_\xi (1 + w),
\]
where $w$ is the enthalpy
\[ w = \Phi + p/\rho_0. \]  
(2.5.37)

The Bernoulli equation (2.5.33) reduces to the familiar form
\[ \Phi + \frac{1}{2} v^2 + w = \text{constant along flow lines.} \]  
(2.5.38)

The Euler equation for a perfect fluid in adiabatic flow, (2.5.30), reduces to
\[ \frac{dv}{dt} = -\nabla \Phi - \frac{1}{\rho_0} \nabla p, \]  
(2.5.39)

where $d/dt$, the derivative with respect to proper time along the flow lines, has the Newtonian form
\[ d/dt = \partial/\partial t + v \cdot \nabla. \]  
(2.5.40)

The first law of thermodynamics, (2.5.12) and (2.5.12'), reduces to
\[ d \Phi = -p dV_0 + T d\Theta_0, \]  
(2.5.41)

\[ d \omega = V_0 dp + T d\Theta_0 \]  
(2.5.42)

(In Newtonian theory one usually adopts a per-unit-mass viewpoint rather than a per-baryon viewpoint; and one uses $\rho_0$, $V_0$, $\Theta_0$ rather than $\rho$, $V$, $\Theta$.)

### 2.6 Radiative Transfer

**Basic references** Appendix I of Paczolczyk (1970); Mihalas (1970); Chandrasekhar (1960); Lindquist (1966).

**Notation and terminology** At an arbitrary event in spacetime pick an arbitrary local Lorentz frame. In that frame pick an arbitrary spatial direction $\textbf{n}$ (n is a unit vector; see Figure 2.6.1). Examine the amount of energy $dE$ that is (i) carried by photons across a unit surface area $dA$ orthogonal to $\textbf{n}$ (the surface $\Sigma_m$ of Figure 2.6.1) during unit time $dt$, with (ii) the photons having frequencies $\nu$ in the range $dv$, and (iii) the photons being directed into a solid angle $d\Omega$ about $\textbf{n}$. The ratio
\[ I_n = \frac{dE}{dt \ dA \ dv \ d\Omega} \]  
(2.6.1)

is called the **specific intensity** or simply the **intensity**. It depends on (a) the event in spacetime; (b) the choice of Lorentz frame; (c) the direction $\textbf{n}$; (d) the frequency $\nu$. One also defines the **total intensity** $I$ by
\[ I = \int I_n \ dv \ d\Omega. \]  
(2.6.2)

the average specific intensity or average intensity $I_n$ (averaged over all directions) by
\[ I_n = \frac{1}{4\pi} \int I_n \ d\Omega; \]  
(2.6.3)

and the **total intensity** $J$ by
\[ J = \frac{1}{4\pi} \int I_n \ dv \ d\Omega. \]  
(2.6.4)

It is easy to see that the **energy density per unit frequency** in the radiation is
\[ \rho^n = \frac{dE}{d^3 x \ dv \ c}, \]  
(2.6.5)

and that the **total energy density** in the radiation is
\[ \rho = \frac{dE}{d^3 x \ c}. \]  
(2.6.6)

Pick a 2-surface $\Sigma_m$ in the chosen Lorentz frame, with unit normal $\textbf{n}$ (Fig. 2.5.1). Examine the total energy $dE$ per unit area $dA$ that is carried across the chosen surface during unit time $dt$, by photons having frequency $\nu$ in the range $dv$. Make no restrictions on the photon direction or solid angle; but count negatively those photons that cross $dA$ from the front side toward the back. The ratio
\[ F_n = \frac{dE}{dt \ dA \ dv} \]  
(2.6.7)

is called the **specific flux**, or simply the **flux**. It is easy to see from Figure 2.5.1 that
\[ F_n = \int I_n \ cos \Theta \ d\Omega. \]  
(2.6.8)

The specific flux depends on (a) the chosen event in spacetime; (b) the chosen Lorentz frame at that event; (c) the chosen 2-surface $\Sigma_m$ in that Lorentz frame.
The integral of the flux over all frequencies is called the total flux

\[ F = \int F_\nu \, d\nu = \int I \cos \theta \, d\Omega. \]  

(2.6.9)

Notice that the total flux is the “first moment” of the total intensity (the “moment” being taken with respect to the unit normal \( m \)). One can readily verify that the “second moment” is equal to the radiation pressure that acts across the surface \( S_m \):

\[ \tau^{(\text{rad})}_{\text{med}} = m \cdot \tau^{(\text{rad})} \cdot m = \int I \cos \theta \, d\Omega. \]  

(2.6.10)

**Invariance of \( I_\nu p^3 \)**

Choose a particular photon that passes through a given event in spacetime. Observe that photon, and all others in the vicinity, from several different Lorentz frames at the given event. The frequency \( \nu \) of the photon will depend on the choice of Lorentz frame (doppler shift from one frame to another). The specific intensity \( I_\nu \) in the neighborhood of the photon will also depend on the choice of Lorentz frame. However, the ratio \( I_\nu p^3 \) will be Lorentz-invariant. (Aside from a factor \( h^{-3} \), \( I_\nu p^3 \) is the invariant number density in phase space

\[ N = \frac{dN}{d^3x d^3p} = \frac{1}{h^3} I_\nu p^3, \]  

(2.6.11)

see, e.g., § 22.6 of MTW.)

When photons propagate freely through curved spacetime (no interaction with matter), the ratio \( I_\nu p^3 \) is conserved along the world line of each photon (Liouville’s theorem).

**Equation of Radiative Transfer**

Consider the propagation of photons through a medium (e.g., through gas that is falling into a black hole). Analyze the photon propagation from the viewpoint of observers at rest in the medium (local rest frame; “LRF”). At each event \( \mathcal{E} \) denote by \( u \) the 4-velocity of the medium—and hence also of the LRF. Focus attention on all photons in the (phase-space) neighborhood of a given null geodesic ray. Denote by \( p \) the 4-momentum of that ray. Then at each event along the given ray \( p \) is given by

\[ p = \hbar \alpha (u + n). \]  

(2.6.12)

Here \( n \) is a unit vector that (i) is purely spatial as seen in the LRF

\[ n \cdot u = 0, \]  

(2.6.13)

and (ii) is interpreted in the LRF as the direction of propagation of the ray (same as 3-vector \( n \) of Figure 2.6.1). Also, in eq. (2.6.12), \( \alpha \) is the frequency of any photon which propagates along the ray, and \( \hbar \alpha \) is its energy, as measured in

the LRF. Because of the acceleration and shear of the medium, the frequency \( \nu \) of the chosen ray changes from point to point along the ray. The change \( d\alpha \) when the ray propagates a proper spatial distance \( dl \) as seen in the LRF, is

\[ d\alpha = -\nabla_u \cdot n \cdot (p \cdot u/h) \, dl. \]

(2.6.14)

This frame-independent equation is derived easily by geometrical arguments in the LRF. A straightforward calculation, using the geodesic equation for the ray

\[ \nabla_u p = \hbar \nu \nabla_u \nu p = 0, \]  

and using expansion (2.5.17) for \( \nabla_u \nu \), reveals

\[ d\alpha/dt = -\nu (n \cdot a + (1/2) \dot{\theta} + n^2 \rho_0^2 \delta_{00}). \]  

(2.6.15)

The first term, \( -\nu n \cdot a \), is the "gravitational redshift" produced by the acceleration of the LRF; the second and third terms, \( -\nu (1/2) \dot{\theta} + n^2 \rho_0^2 \delta_{00} \), are the "cosmological redshift" due to the expansion of the medium along the direction of the ray.

If there were no interaction with the medium, then \( I_\nu p^3 \) would be conserved along the ray. Hence, the equation of radiative transfer along the given ray must have the form

\[ dI_\nu/d\nu = -\frac{3}{\nu} \frac{d\alpha}{dt} I_\nu = \text{(effects of interaction with medium)}; \]

i.e.,

\[ dI_\nu/d\nu + (3n \cdot a + \dot{\theta} + \rho_0^2 \delta_{00}) I_\nu = \text{(interaction effects).} \]  

(2.6.16)

Four types of interaction can occur: spontaneous emission of radiation by the matter; stimulated emission; absorption; and scattering.

We shall assume that on all length scales of interest the medium is isotropic; and we shall denote its emissivity by

\[ \epsilon_\nu = \frac{dI_\nu}{d\nu \, dt \, dm_0} \text{ (energy emitted spontaneously per unit frequency during unit time by a unit rest mass, integrated over all angles, and measured in LRF).} \]  

(2.6.17)

(The emissivities for free-free and free-bound transitions were discussed in §§ 2.1 and 2.2.) Then spontaneous emission contributes

\[ \frac{dI_\nu}{dt} \text{ (spontaneous emission) = } \frac{1}{4\pi} \rho_0 \epsilon_\nu \]  

(2.6.18)

to the specific intensity along the given ray. Here \( \rho_0 \) is the density of rest mass in the medium.
The rate of absorption and the rate of stimulated emission are both proportional to the intensity of the passing beam. Hence, the effects of absorption and of stimulated emission can be lumped together into a single absorption coefficient $\kappa_\nu$, defined by

$$\left( \frac{dl_\nu}{dt} \right)_{\text{absorption plus stimulated emission}} = -\rho_0 \kappa_\nu I_\nu. \quad (2.6.18)$$

The dimensions of the absorption coefficient are cm$^2$/g ("absorption cross section per unit mass" at the given frequency). When absorption dominates over stimulated emission, $\kappa_\nu$ is positive; when stimulated emission dominates, $\kappa_\nu$ is negative ("negative absorption"). Scattering is more complicated to treat than emission and absorption. Let

$$\frac{d\kappa_\nu}{ dl_\nu} (p, p')$$

be the differential cross section (as measured in the LRF) for a unit rest mass to convert a photon of momentum $p$ into a photon of momentum $p'$. Then let

$$\kappa_\nu(p) = \int \frac{d\kappa_\nu}{ dl_\nu} (p, p') dl_\nu' \quad (2.6.19)$$

be the total scattering cross-section ("scattering opacity") per unit rest mass.

For example, in the case of electron scattering by a nonrelativistic plasma ($hv < mc^2$, $kT < mc^2$; see §4.4), when one neglects the tiny change in photon frequency the cross sections per unit mass are

$$\frac{d\kappa_\nu}{ dl_\nu} (p, p') = \frac{f_\nu}{m_e^2} \frac{1}{r_\nu^2} \left[ 1 + \left( n - n^* \right)^2 \right] \delta(\nu' - \nu)$$

$$\kappa_\nu = (f_\nu/m_e^2)(8\pi^3/3) r_\nu^2 = (0.40 \text{ cm}^2/\text{g}) f_\nu \quad (2.6.20)$$

(Recall: $n = p/hv$, $n^* = p'/hv$; $f_\nu$ is the number of free electrons per baryon in the plasma.) Scattering, as described by the appropriate differential cross section, can increase $I_\nu$ (scattering into the beam from other directions), or can decrease $I_\nu$ (scattering out of the beam to other directions):

$$\left( \frac{dl_\nu}{dt} \right)_{\text{scattering}} = + \rho_0 \int \frac{d\kappa_\nu}{ dl_\nu} (p', p') dl_\nu' \quad (2.6.21)$$

By combining equations (2.6.15)-(2.6.21), we obtain our final form of the equation of radiative transfer

$$\frac{dl_\nu}{dl} + (3n \cdot a + \theta + 3\pi a^2 \sigma_{\text{tot}}) = (4\pi)^{-1} \rho_0 \kappa_\nu - \rho_0 \kappa_\nu I_\nu$$

$$+ \rho_0 \int \frac{d\kappa_\nu}{ dl_\nu} (p', p) dl_\nu' \quad (2.6.22)$$

Relationship between emissivity and absorption

Into an insulated box place a material medium and a radiation field; and then wait until complete thermodynamic equilibrium is achieved. If $T$ is the equilibrium temperature, then the radiation field as seen in the LRF will be isotropic with the standard black-body intensity, $I_\nu = B_\nu$, where

$$B_\nu = \frac{(2h c^2 \nu^3)}{e^{h\nu/kT} - 1} \quad (2.6.23)$$

In equilibrium there must be no net change of $I_\nu$ along any ray: scattering into the beam must be completely balanced by scattering out of the beam; and emission must be completely balanced by absorption. The requirement of "detailed balance" for emission and absorption can be met only if the emissivity and the absorption coefficient are related by

$$\frac{dl_\nu}{dl} = \rho_0 \kappa_\nu \left( \frac{e_\nu}{4\pi \kappa_\nu} - I_\nu \right) = \rho_0 \kappa_\nu \left( \frac{e_\nu}{4\pi \kappa_\nu} - B_\nu \right) = 0; \quad (2.6.24)$$

i.e.,

$$e_\nu/4\pi \kappa_\nu = B_\nu. \quad (2.6.24)$$

Since $e_\nu$ and $\kappa_\nu$ depend only on the thermodynamic state of the matter, and have nothing to do with the state of the radiation, relation (2.6.24) must be satisfied not only inside our insulated box, but also in all other cases where the matter by itself is in thermodynamic equilibrium but the radiation might not be.

For matter not in thermodynamic equilibrium one can use a more sophisticated version of the principle of detailed balance ("Einstein A and B coefficients") to derive a more complicated relationship between the emissivity and the absorption coefficient. See, e.g., Chandrasekhar (1960).

As an application of the equilibrium relationship $e_\nu/4\pi \kappa_\nu = B_\nu$, consider free-free transitions in an ionized gas. Whenever the free electrons (which do the emitting and absorbing) are in thermodynamic equilibrium with each other (Maxwell-Boltzmann velocity distribution), the absorption coefficient— as derived from the emissivity (2.1.27)— must be

$$\kappa_\nu^\pi = (1.50 \times 10^{24} \text{ cm}^2/\text{g}) \left( \frac{f_\nu^\pi}{A} \right) \left( \frac{\rho_0}{8 \text{ cm}^{-3}} \right) \nu^{7/2} \sigma T_\nu^{\gamma/2} \left( \frac{1 - e^{-\tau}}{x^3} \right) \quad (2.6.25)$$

$x \equiv h\nu/kT$.

See §2.1 for notation.
Notice that for matter in thermodynamic equilibrium the relationship
\[ \varepsilon_{\nu} / 4\pi \kappa_{\nu} = B_{\nu} \] permits one to rewrite the equation of radiative transfer (2.6.22) in the form
\[ \frac{dI_{\nu}}{dl} + (3n \cdot a + \theta + 3n^2a^2_\Theta) = \rho_0 \kappa_{\nu} (B_{\nu} - I_{\nu}) + \frac{dI_{\nu}}{dl}_{\text{scattering}}. \] (2.6.26)

or, equivalently,
\[ \frac{dI_{\nu}(\nu^2)}{dl} = \rho_0 \kappa_{\nu} \left[ \frac{B_{\nu}}{\nu^5} - \frac{I_{\nu}}{\nu^5} \right] + \frac{dI_{\nu}(\nu^2)}{dl}_{\text{scattering}}. \] (2.6.27)

**Optical depth**

Consider radiation propagating out of a medium into surrounding empty space. Follow a given ray “backward”, from “infinity” into the medium. Let the ray have frequency \( \nu_{\infty} \) at infinity; then its frequency at location \( l \) (l = proper distance measured in LRF of medium) is
\[ \nu(l) = \nu_{\infty} \exp \left( -\int (n \cdot a + \frac{1}{2} \theta + n^2a^2_\Theta) dl \right). \] (2.6.28)

[cf. eq. (2.6.14)]. In calculating the change of intensity along the ray, it is often useful to replace the proper-length parameter \( l \) by the “optical-depth” parameter
\[ \tau_{\nu} = \int \rho_0 \kappa_{\nu} dl. \] (2.6.29)

In the integration \( \kappa_{\nu} \) must be evaluated at the frequency \( \nu(l) \). The optical depth \( \tau_{\nu} \) depends on (i) the world line of the ray in spacetime; (ii) location along that world line; and (iii) the frequency of the ray at “infinity”, \( \nu_{\infty} \).

One can also introduce an optical depth for scattering radiation out of the beam, \( \tau_{\nu}^{(s)} \):
\[ \tau_{\nu}^{(s)} = \int \rho_0 \kappa_{\nu}^{s} dl. \] (2.6.30)

In terms of optical depths, the laws of radiative transfer (2.6.27) reads
\[ -\frac{dI_{\nu}(\nu^2)}{d\tau_{\nu}} = \frac{B_{\nu}}{\nu^5} - \frac{I_{\nu}}{\nu^5} + \frac{dI_{\nu}(\nu^2)}{d\tau_{\nu}}_{\text{scattering}}, \] (2.6.31)

where
\[ \frac{dI_{\nu}(\nu^2)}{d\tau_{\nu}}_{\text{scattering}} = -\kappa_{\nu} \left[ \frac{1}{\kappa_{\nu}^{s}} - \frac{1}{\kappa_{\nu}^{s}} \right] \int \frac{d\tau_{\nu}}{d\nu} \left( \nu', \nu \right) I_{\nu'} dl' \] (2.6.32)

One says that a medium is **optically thick** to emission and absorption (or to scattering) along a given ray if optical depths \( \tau_{\nu} \gg 1 \) (or \( \tau_{\nu}^{(s)} \gg 1 \)) are achieved along the ray. One says that the medium is **optically thin** if everywhere along the ray \( \tau_{\nu} \ll 1 \) (or \( \tau_{\nu}^{(s)} \ll 1 \)).

Consider a medium that (i) has its matter in thermodynamic equilibrium with a spatially uniform temperature, (ii) is optically thick along a chosen ray, and (iii) has emission and absorption dominant over scattering, i.e.,
\[ \kappa_{\nu} \gg \kappa_{\nu}^{s}. \] (2.6.33)

along that ray. Then the radiation emerging to infinity along the chosen ray must have the blackbody form
\[ I_{\nu} = \frac{B_{\nu}}{\nu^5} = \frac{2h(c^4)}{e^{h/(kT)} - 1}. \] (2.6.34)

[One can prove this easily from the equation of transfer (2.6.31).] Moreover, at a point in the medium where all rays are optically thick, the radiation will have a blackbody intensity at all frequencies and in all directions; so the energy density and pressure in the radiation will be
\[ u^\nu = p^\nu = bT^4, \] (2.6.35)

where \( b \) is the universal constant
\[ b = \frac{8\pi^5k^4}{15c^3h^3} = 7.56 \times 10^{-15} \text{ ergs} / \text{cm}^3 \text{K}^4. \] (2.6.36)

Consider, alternatively, a medium that is optically thin along a chosen ray, and has negligible scattering (\( \tau_{\nu}^{(s)} \ll 1 \)) along that ray. Then the equation of transfer (2.6.31) predicts for the radiation emerging to infinity
\[ I_{\nu}(\nu^2) = \int_{\text{entire ray}} (B_{\nu}(\nu^2)) dl_{\nu} = (1/4\pi) \int (e_{\nu}(\nu^2)) \rho_0 dl \] (2.6.37)

\[ \approx \left( \frac{\rho_0 dl}{4\pi} \right) (1/4\pi) (e_{\nu}(\nu^2)) \text{in region of strongest emission}. \] (2.6.38)

**Radiative transfer in the diffusion approximation**

Consider the interior of an optically thick medium \( \tau_{\nu} \gg 1 \) which is in local thermodynamic equilibrium. Suppose that the medium has a temperature
gradient and/or an acceleration, but that the characteristic length scale

$$l_T = \frac{T}{|\nabla T + \mathbf{a_T}|} \quad (2.6.39)$$

over which the temperature changes are long compared to the mean-free path of a photon,

$$l_p = 1/\kappa_\nu \rho_0 \ll l_T. \quad (2.6.40)$$

Then the radiation distribution will consist of a large, isotropic, blackbody component, plus a tiny "correction" due to the temperature gradient

$$I_\nu = B_\nu + I_\nu^{(1)}, \quad I^{(1)}_\nu \ll B_\nu. \quad (2.6.41)$$

There is no net flux $F$ across any surface associated with the blackbody component $B_\nu$; but the "correction" term $I^{(1)}_\nu$ will lead to a flux in the direction of

$$\mathbf{h} \cdot (\nabla T + a_T). \quad (2.6.42)$$

(Here $h$ is the projection operator of §2.5.) One can calculate the magnitude of this flux by inserting expression (2.6.41) for $I_\nu$ into the equation of transfer (2.6.22), and by then integrating over $\cos \theta \, d\Omega \, dv$. (Here $\theta$ is the angle between $\mathbf{h} \cdot (\nabla T + a_T)$ and the direction of d$\Omega$.) The result is

$$q = \frac{1}{\kappa_\nu \rho_0} \left( \frac{2}{3} \right) \mathbf{h} \cdot (\nabla T + a_T). \quad (2.6.43)$$

Here $q$ is the energy flux vector of §2.5, and its magnitude is the flux of energy in the $\mathbf{h} \cdot (\nabla T + a_T)$ direction

$$\|q\| = F. \quad (2.6.44)$$

Also, in eq. (2.6.43) $\kappa$ is the "Rosslan mean opacity", defined by

$$\frac{1}{\kappa} = \int_\nu \frac{(\kappa_\nu + \kappa_\sigma)^{-1} \, (dB_\nu/dT) \, dv}{\int_\nu (dB_\nu/dT) \, dv}. \quad (2.6.45)$$

For free-free transitions by themselves (eq. 2.6.25) the Rosseland mean opacity is

$$\kappa_{ff} = (0.645 \times 10^{23} \text{ cm}^{-2} \text{ g}^{-1}) \left( \frac{c_\sigma \nu_f Z^2}{A} \right) \left( \frac{\rho_0}{\text{ g cm}^{-3}} \right) T_k^{-7/2}. \quad (2.6.46)$$

See §2.1 for notation. For electron scattering by itself in the nonrelativistic case, the Rosseland mean opacity is

$$\kappa_{ee} = \kappa_{ee} = 0.4 \text{ cm}^{-1} \text{ g}^{-1}. \quad (2.6.47)$$

2.7 Shock Waves

Basic references: Zel’’dovich and Raizer (1966); Landau and Lifshitz (1959); Taub (1948); Lichnerowicz (1967, 1970, 1971); Thorne (1973a).

Types of shock waves

When gas in supersonic flow encounters an obstacle (e.g., the surface of a star, or the geometric structure of the ergosphere of a black hole), a shock front develops. In the shock front the flow decelerates sharply from supersonic to subsonic, and some of the kinetic energy of the flow gets converted into heat (increase in entropy).

The structure of the shock depends on the nature of the forces which decelerate the gas particles (atoms, ions, electrons). If those forces are collisions between the particles themselves (the usual case), one has an ordinary shock. But if the particles are decelerated without colliding—e.g., by impact onto the dipole magnetic field of a neutron star, which swings the particles into Larmor orbits—one has a collisionless shock. These notes will be confined to ordinary shocks. For the theory of collisionless shocks see, e.g., the end of §12 of Kaplan (1966).

Shock waves in a partially ionized gas (e.g., the interstellar medium) can have a somewhat different form than ordinary shocks. As gas passes through the shock front, much of its kinetic energy of supersonic flow can be converted into excitation energy of atoms and ions, and can be quickly radiated away as "line" radiation. The result is a bright glow from the shock front itself. See, e.g., Pikel’ner (1961) and §10 of Kaplan (1966). In this section we shall ignore such energy losses to radiation—i.e., we shall demand that the gas behind the shock have the same total energy per unit rest mass as the gas in front of the shock.

The relativistic Rankine-Hugoniot equations

Pick a particular event $\mathcal{P}$ on a shock front. In the neighborhood of $\mathcal{P}$ introduce a local Lorentz frame ("rest frame of the shock") in which (i) the shock is momentarily at rest; (ii) the shock is the surface $y = x = 0$; and (iii) on both sides of the shock the fluid is moving in the $x$ direction, i.e., perpendicular to the shock front ("normal shock"); see Figure 2.7.1.) That such a local Lorentz frame exists in general one can prove quite easily [see, e.g., Taub (1948)].

Denote the "front" side of the shock (side from which the fluid moves) by a "$1$", and denote the "back" side (side toward which the fluid moves) by a "$2$"; see Figure 2.7.1. Denote the velocity of the fluid, as measured in the rest frame of the shock, by

$$v_1 = (dx/dt)_1 = \text{ordinary velocity on front side}, \quad (2.7.1a)$$

$$v_2 = (dx/dt)_2 = \text{ordinary velocity on back side}, \quad (2.7.1b)$$
\[ u_1 = v_1 \gamma_1 = v_1 / (1 - v_1^2)^{1/2} \] \text{"4-velocity" on front side,} \tag{2.7.1c}

\[ u_2 = v_2 \gamma_2 = v_2 / (1 - v_2^2)^{1/2} \] \text{"4-velocity" on back side.} \tag{2.7.1d}

(Note that these "4-velocities" are scalars, not vectors.)

In the rest frame of the shock the law of baryon conservation is equivalent to continuity of the baryon flux:

\[ j = n_1 u_1 = n_2 u_2, \tag{2.7.2a} \]

\[ \text{FRONT ("1" side)} \]
\[ \text{BACK ("2" side)} \]

\[ v_1 \]
\[ v_2 \]
\[ \beta \]
\[ \gamma \]
\[ \theta \]
\[ \phi \]

Figure 2.7.1. A shock wave viewed in the local-Lorentz rest frame of the shock front at an event \( \theta \).

Similarly, energy and momentum conservation are equivalent to continuity of energy flux and momentum flux:

\[ (\rho + p)u_1 = (\rho + p)u_2, \tag{2.7.2b} \]

\[ (\rho + p)u_1^2 + p_1 = (\rho + p)u_2^2 + p_2. \tag{2.7.2c} \]

These junction conditions are more easily understood by writing them in a form analogous to the Rankine-Hugoniot equations of Newtonian theory: First take the law of baryon conservation (2.7.2a) and turn it into equations for the fluid 4-velocities

\[ u_1 = jV_1, \quad u_2 = jV_2. \tag{2.7.3a} \]

(See §2.5 for notation.) Then take the law of momentum conservation (2.7.2c); rewrite it in terms of \( \mu, V, j, \) and \( p \) using the law of baryon conservation (2.7.2a) and the relations \( n = 1/V, \mu = (\rho + p)/V \); and solve for the baryon flux to obtain

\[ j^2 = \frac{p_2 - p_1}{\mu_2 V_2 - \mu_1 V_1}. \tag{2.7.3b} \]

Finally, use the law of baryon conservation (2.7.2a) to rewrite the energy equation (2.7.2b) in the form

\[ \mu_1 \gamma_1 = \mu_2 \gamma_2. \]

Divide equation (2.7.3b) by \( (\mu_1 V_1 + \mu_2 V_2) \), and combine with \( j = u_1/V_1 = u_2/V_2 \) [eq. (2.7.2a)] to obtain

\[ \mu_2 V_2^2 - \mu_1 V_1^2 = (p_1 - p_2)(\mu_1 V_1 + \mu_2 V_2); \]

then subtract this from the square of the energy equation \( (\mu_1 \gamma_1)^2 - (\mu_2 \gamma_2)^2 = 0 \), to obtain

\[ \mu_2^2 - \mu_1^2 = (p_2 - p_1)(\mu_1 V_1 + \mu_2 V_2). \tag{2.7.3c} \]

Equations (2.7.3) are Taub's (1948) junction conditions for shock waves. Their Newtonian limits are the standard Rankine-Hugoniot equations:

\[ v_1 = j_0 V_{01}, v_2 = j_0 V_{02} \tag{2.7.4a} \]

(\( j_0 \) is "mass flux" of Newtonian theory);\n
\[ \frac{j_0^2}{\rho_0} = \frac{p_2 - p_1}{V_{02} - V_{01}}. \tag{2.7.4b} \]

\[ w_2 - w_1 = \frac{1}{2}(p_2 - p_1)(V_{01} + V_{02}). \tag{2.7.4c} \]

The form of the Newtonian junction conditions (2.7.4) motivates one to use \( \rho \) and \( V_0 \) as one's independent thermodynamic variables when analyzing Newtonian shocks; similarly, the form of the relativistic junction conditions (2.7.3) motivates one to use \( p \) and \( \mu V \) as one's independent variables for relativistic shocks.

The Rankine-Hugoniot curve

Consider a family of shocks, each with the same thermodynamic state on the front face (same \( \mu_1, V_1, p_1 \), etc.), but with different states on the back face (different \( \mu_2 V_2, p_2 \), etc.). This family of shocks is a one-parameter family. Thus, if one plots all back-face states \( (\mu_2 V_2, p_2) \), in the \( \mu V - p \) plane, they lie on a single curve—the "Rankine-Hugoniot curve"—passing through the point \( (\mu_1 V_1, p_1) \). (In the relativistic case this curve is also called the "Taub adiabat".)

One can also plot, in the \( \mu V - p \) plane, the Poisson adiabat (curve of constant entropy) passing through \( (\mu_1 V_1, p_1) \). These two curves typically have the relative shapes and locations shown in Figure 4.7.2. In particular, from the Rankine-Hugoniot equations one can derive the following general properties of the Rankine-Hugoniot
(i) The Rankine-Hugoniot curve is tangent to the Poisson adiabat, and has the same second derivative at point “1”; i.e., for weak shocks the increase in entropy is third-order in the pressure jump:

\[
s_2 - s_1 = \frac{1}{12x'} \frac{\partial^2 (\mu V)}{\partial p^2} \mu_1 (p_2 - p_1)^3 + 0[(p_2 - p_1)^4].
\]  

(2.7.5)

(ii) As the gas passes from the front of the shock to the back, its entropy, pressure, and chemical potential increase

\[
s_2 > s_1, \quad p_2 > p_1, \quad \mu_2 > \mu_1.
\]  

(2.7.6a)

\[\text{Figure 2.7.2. The Poisson adiabat } \mathcal{P} \text{ and the Rankine-Hugoniot curve } \mathcal{H} \text{ plotted in the } \mu V p \text{ plane.}\]

while its specific volume and the product } \mu V \text{ decrease

\[
V_2 < V_1, \quad \mu_2 V_2 < \mu_1 V_1.
\]  

(2.7.6b)

(Hence, only the “upper branch” of the Rankine-Hugoniot curve, Figure 2.7.2, is physically relevant.)

(iii) The flow on the front side is always supersonic; that on the back side is always subsonic

\[
u_2/\nu_1 > 1, \quad \nu_2/\nu_2 < 1;
\]  

\[
\nu_2 = c_2/(1 - c_2^2)^{1/2}
\]  

(2.7.7)

† The derivation requires the assumption

\[
|\partial^2 (\mu V)/\partial p^2|_1 > 0;
\]

a condition which is satisfied by all materials of astrophysical or laboratory interest; see Landau and Lifshitz (1959). Some, but not all, of the listed properties remain true without this assumption; see, e.g., Zel’dovich and Raizer (1966) for nonrelativistic details, and Thorne (1973a) for relativistic details.

(iv) As one moves up the Rankine-Hugoniot curve, away from point “1”—i.e., as one studies a sequence of ever “stronger” shocks—the following quantities increase monotonically:

- the baryon flux across the shock, \( j \);
- the jump in entropy across the shock, \( s_2 - s_1 \);
- the “relativistic” mass number on the front side of the shock, \( M_1 = u_1/\mu_1 \).

Notice that, once the thermodynamic state on the front face of the shock has been specified, the shock has only one free parameter. If one fixes the baryon flux across the shock \( j \), or the speed on the front face \( u_1 \), or the pressure on the back face \( p_2 \), or any other single parameter, then all other properties of the shock are uniquely determined. To calculate them, one need merely invoke the Rankine-Hugoniot equations (2.7.4), the laws of thermodynamics, and the equation of state of the gas.

**Shocks in an ideal gas**

Consider, as a special but important case, a nonrelativistic ideal gas with constant adiabatic index \( \Gamma = -\partial \ln p/\partial \ln V \), and with mean rest mass per particle \( \bar{m} \), so that

\[
pV_0 = kT/\bar{m}.
\]  

(2.7.9)

By a straightforward calculation (§85 of Landau and Lifshitz 1959), one can derive the following relationships between various quantities along the Rankine-Hugoniot curve:

\[
V_{02} = \frac{(\Gamma + 1)p_1 + (\Gamma - 1)p_2}{\Gamma - 1} \quad \text{for strong shock,}
\]

\[
V_{01} = \frac{(\Gamma - 1)p_1 + (\Gamma + 1)p_2}{\Gamma + 1} \quad \text{for strong shock,}
\]

\[
T_2 = T_1 \left( \frac{\Gamma + 1}{\Gamma - 1} \right) \frac{p_2}{p_1} \quad \text{for strong shock,}
\]

\[
\frac{1}{\Gamma - 1} \frac{p_2 + p_1}{p_1 + p_2} \frac{1}{2V_{01}} \quad \text{for strong shock,}
\]

\[
J_2 = J_1 \left( \frac{\Gamma + 1}{\Gamma - 1} \right) \frac{p_2}{p_1} \quad \text{for strong shock,}
\]

\[
\frac{1}{\Gamma - 1} \frac{p_2 + p_1}{p_1 + p_2} \frac{1}{2V_{01}} \quad \text{for strong shock.}
\]  

(2.7.10)

Here “strong shock” means “in the limit \( p_2/p_1 \gg 1 \).”

**2.8 Turbulence**

In the accretion of gas onto a black hole, turbulence probably plays an important role. (See §5.4 and 5.) Unfortunately, the theory of turbulence is in a very
uncertain state. Little is known with confidence about the astrophysical circumstances under which turbulence should develop, or about the strength of the turbulence in various situations. For overviews of the current state of one's knowledge see, e.g., Chapter 3 of Landau and Lifshitz (1959); also Pikel'n er (1961).

2.9 Reconnection of Magnetic Field Lines


Basic ideas

In flat spacetime consider a plasma that is macroscopically neutral, that has a high electrical conductivity \( \kappa \), and that is sufficiently dilute for one to ignore its dielectric properties: \( \varepsilon = \mu = 1 \). Let the plasma be endowed with a large-scale magnetic field, and examine the evolution of that field in a Lorentz frame where the plasma has low velocity \( |v| \ll c \). Maxwell's equations for the electromagnetic field then read

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 0, \\
\nabla \times \mathbf{E} + (1/c)(\partial \mathbf{B}/\partial t) &= 0, \\
\nabla \times \mathbf{B} - (1/c)(\partial \mathbf{E}/\partial t) &= 4\pi \mathbf{j}.
\end{align*}
\]

(2.9.1)

In the local rest frame of the plasma the current is proportional to the electric field, \( \mathbf{J} = \sigma \mathbf{E} \). When transformed to the Lorentz frame where the plasma moves with velocity \( \mathbf{v} \), this equation says

\[
\mathbf{J} = \sigma (\mathbf{E} + (v/c) \times \mathbf{B}).
\]

(2.9.2)

A straightforward calculation from the above equations leads to the following law for the rate of change of magnetic field along the world lines of the plasma:

\[
dB/d\tau = (\omega + \sigma - \frac{3}{2} \theta) \cdot \mathbf{B} - \frac{1}{4\pi c} \left( \frac{\partial^2 \mathbf{B}}{\partial t^2} - c^2 \nabla^2 \mathbf{B} \right).
\]

(2.9.3)

Here \( \omega \) is the unit 3-tensor; \( \omega, \sigma, \) and \( \theta \) are the rotation, shear, and expansion of the plasma [cf. eqs. (25.17) and (25.18)]; and

\[
d/d\tau = \beta \partial t + v \cdot \nabla.
\]

(2.9.4)

Because of the very high electrical conductivity, one can ignore the "wave-equation" part of the evolution law (2.9.3) almost everywhere in the plasma; and the magnetic field evolves in a "frozen-in" manner:

\[
 dB/d\tau = (\omega + \sigma - \frac{3}{2} \theta) \cdot \mathbf{B}.
\]

(2.9.5)

[cf. Eq. (25.20) and associated discussion.] However, in regions of plasma where the field has strong gradients, the "wave-equation" part must come into play.

Consider, as the case of greatest importance, a magnetic field that is chaotic with an ordered structure on some scale \( l_c \) (subscript \( c \) for "cell size"). At the

Figure 2.9.1. The region in a plasma where magnetic field lines are reconnecting (schematic picture). In the top view, (a), the reconnection region is stippled; the magnetic field lines are drawn solidly; and the flow lines of the plasma are dotted. The innermost field line is in the process of reconnecting into the dashed form. The perspective view, (b), shows the closed curve \( \gamma \) around which one integrates to derive eq. (2.9.6) for the EMF in the connecting region.

interface between adjacent cells, the magnetic field must reverse sign ("neutral point" or "neutral sheet"), and its gradients will be very large. Hence, at the interface, "wave-equation" behavior can come into play destroying the frozen-in behavior of the field. The result is a reconnection of magnetic field lines, as
depicted in Figure 2.9.1. This reconnection gradually converts the two adjacent cells into a single cell, reducing the overall strength of the magnetic field.

In situations of interest (§§4.4 and 5.2), the decrease of field strength due to reconnection will be counterbalanced by an increase in strength due to compression or shear of the plasma.

The reconnection process creates very strong electric fields. In particular, a simple application of Faraday's law

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A},$$

with the curve $\Gamma$ as shown in Figure 2.9.1b, shows that the EMF built up in the reconnecting region is

$$\Delta \Phi = \text{(rate of reconnection of magnetic flux)} \equiv \frac{d\Phi}{dt}. \quad (2.9.6)$$

In situations of interest to us [§5.12; Lynden-Bell (1969)], these EMFs can be far greater than $10^{16}$ volts. If the density of the plasma is sufficiently low to permit long mean-free paths for charged particles, and if the reconnecting region is sufficiently straight, then these EMFs can accelerate particles to ultra-relativistic energies; and the particles can then radiate intense synchrotron radiation. In this manner, regions of reconnecting field lines may be sources of flares.

3 The Origin of Stellar Black Holes


Basic ideas and issues

How much of the mass of the Galaxy is in the form of black holes? What is the spectrum of black-hole masses? How many black holes are created per year by stellar collapse in our Galaxy? To these questions one has only the vaguest of answers in 1972.

If the spectrum of stellar masses at birth is the same elsewhere as in the solar neighborhood, and always has been the same, then roughly half of the mass of the Galaxy has been cycled through stars of $M > 2M_\odot$, and such stars are being born at a rate of about 0.2 per year. If the maximum mass of a neutron star is $2M_\odot$, and if stars of $M > 2M_\odot$ lose a negligible amount of mass during their evolutions and deaths, then all of the matter which goes into such stars eventually winds up in black holes. Thus, 5 per cent of the mass of the Galaxy might be in black holes, and new black holes might be forming at a rate of $\sim 0.2$ per year. Might. But very probably not, because the if’s which go into this result are probably not satisfied.
In particular, both observation and theory suggest that mass loss is very significant during the late stages of stellar evolution and during the death throes of massive stars. Mass loss might be so important that black holes are rare beasts, indeed!

Let us focus attention on mass loss during the death throes, as predicted by the best numerical calculations to date. But in doing so, let us keep in mind the very primitive state of the modern computations—for example, their typical neglect of the effects of rotation.

Modern computations conclude that stars whose masses exceed the Chandrasekhar limit, \( M > 1.4 M_\odot \), at the endpoint of their evolution should explode as supernovae. A supernova explosion can leave behind two remnants: an expanding gas cloud (the outer part of the original star), and a remnant "star".

The remnant "star" will be a neutron star if its mass, \( M_{\text{remnant}} \), is less than \( \approx 2 M_\odot \); but will be a black hole if its mass is greater than \( \approx 2 M_\odot \).

We summarize in Table 3.1 the results of various calculations of the supernova process.

In Table 3.1 \( M_{\text{core}} \) is formally the mass of the entire star used in the numerical calculations. However, all the calculations assumed a homogeneous initial model (typically polytropic or isothermal), whereas the theory of stellar evolution predicts a highly inhomogeneous structure for presupernova stars. In particular, stellar evolution predicts a central core of iron which is more or less homogeneous, surrounded by a huge diffuse envelope of oxygen, helium, hydrogen, and other elements. Thus, we must imagine the initial stars of the supernova calculations to be the central, homogeneous core; and we must regard those calculations as neglecting the envelope. Such neglect is reasonable when one studies supernova dynamics, because the envelope is so large that it has not enough time to do anything while the core is collapsing and reexpanding. The envelope will generally be thrown off, with little residue of work, by the reexpanding parts of the core. The mass of the envelope might be twice that of the core—or perhaps only the same as the core, or perhaps even less. The theory of stellar evolution is far from definitive on this point. Hence the question marks in column 2 of Table 3.1.

It is clear, from the diverse results shown in Table 3.1, that the theory of supernova explosions is far from perfect in 1972. Nevertheless, one can form the following very tentative conclusions. (i) For stars which approach the ends of their quasistatic evolution with masses less than \( \approx 12 \) to \( 30 M_\odot \), the supernova explosion may produce a neutron star. (ii) For stars with masses greater than \( \approx 12 \) to \( 30 M_\odot \), the explosion may produce a black hole. If this tentative conclusion is correct, then no more than \( \approx 1 \) per cent of the mass of the Galaxy should be in the form of black holes today; and new black holes should be created at a rate no greater than \( \approx 0.01 \) per year.

---

1 Here one should not ignore the large mass defect for massive neutron stars; see Zel’dovich and Novikov (1971).

4 Black Holes in the Interstellar Medium

4.1 Accretion of Noninteracting Particles onto a Nonmoving Black Hole

Consider a black hole at rest in the interstellar medium; and temporarily treat the interstellar gas as though it were made up of noninteracting particles [collisions neglected; mean free paths large compared to the region over which the hole’s gravity makes itself felt, \( r_p \gg 2GM/v^2 \); see below]. In the case of a Schwarzschild hole, all particles with angular momentum per unit mass \( L < 2\pi \rho c \) eventually get captured by the hole (see Box 25.6 of MTW or Figure 12 of ZN). Here \( r_g = 2GM/c^2 \) is the gravitational radius of hole and \( M \) is the hole’s mass. If the particle speeds far from the hole are \( v_m \), then this condition for capture corresponds to an impact parameter

\[
b = L/v_m < b_{\text{capture}} = 2\pi\rho c/v_m.
\]

(4.1.1)

Consequently, the “capture cross section” of the hole is

\[
\sigma = \pi(b_{\text{capture}})^2 = 4\pi\rho^2 c/v_m.
\]

(4.1.2)

For a Kerr hole the capture cross section is of this same order of magnitude. The rest mass per unit time crossing inward through a sphere of \( r \gg b_{\text{capture}} \), with particles directed into the capture region (solid angle \( \Delta \Omega = o/r^2 \)), is

\[
M_0 = \frac{\rho \Omega v_m c}{4\pi} \Delta \Omega = \rho \Omega v_m c = 4\pi\rho^2 c v_m.
\]

(4.1.3)

This is the rate at which the hole accretes rest mass. Rewritten in more astronomical units, this accretion rate is

\[
\frac{d(M_0/M_\odot)}{dt(10^{10} \text{ yrs})} = 10^{-13} \frac{\rho_m}{10^{-24} \text{ g cm}^{-3}} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{v_m}{10 \text{ km sec}^{-1}}\right)^{-1}.
\]

(4.1.3')

The typical densities and speeds of interstellar gas particles in ionized, "H II" regions are \( 10^{-24} \text{ g cm}^{-3} \) and \( 10 \text{ km sec}^{-1} \); see e.g. Kaplan and Pikel’ner (1970). Even if 100 per cent of the infalling rest mass were somehow converted into outgoing radiation, the total luminosity would be only

\[
L_{\text{max}} = \frac{10^{24} \text{ erg sec}^{-1}}{10^{-24} \text{ g cm}^{-3}} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{v_m}{10 \text{ km sec}^{-1}}\right)
\]

(4.1.4)

—a value much too small to be astronomically interesting. Therefore it is fortunate that the electrons, ions, and magnetic fields of the interstellar gas interact strongly enough to make the accretion process obey fluid-dynamic laws rather than the laws of noninteracting particles (see §4.6, below).

† Material for this section is drawn largely from Chapter 13 of ZN, and from Schwartzman (1971).
4.2 Adiabatic, Hydrodynamically Accretion onto a Nonscoring Back Hole

Switching from the relativistic description of accretion to the hydrodynamic
one, we can use the fact that the accretion rate is given by

\[ \dot{M} = \frac{4\pi G M}{c^3} \left( \frac{u}{c} \right)^2 \]

(4.2.4)

The only unknown parameter in the coupled equations (4.2.3) and (4.2.4) is the
mass of the central object, which is often considered to be a constant.

Rewrite the law of mass conservation (4.1.2) with density, \( \rho \), expressed in terms of
sound speed, \( c \).

\[ \rho = \frac{m}{V} \]

(4.2.5)

The mass flux, \( \dot{M} \), is constant, independent of \( r \).

The equation of state of the gas is given by

\[ P = \frac{\rho}{c^2} \]

(4.2.6)

where \( P \) is the pressure and \( c \) is the sound speed.

The flow of gas is governed by the gravitational potential energy and
the hydrostatic stress tensor. The gravitational potential energy is
proportional to the mass and the sound speed, and the hydrostatic
stress tensor is proportional to the density and the sound speed.

The two fundamental equations of conservation of mass and momentum
are

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} = 0 \]

(4.2.7)

and

\[ \frac{\partial \rho v}{\partial t} + \frac{\partial \rho v^2}{\partial r} + \frac{\partial \rho v P \rho}{r} = 0 \]

(4.2.8)

respectively.

The solution of these equations gives the density and velocity profiles
of the accreting gas. The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.9)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.10)

The mass flux is given by

\[ \dot{M} = \frac{4\pi G M}{c^3} \left( \frac{u}{c} \right)^2 \]

(4.2.11)

where \( u \) is the velocity of the gas relative to the black hole.

The equations (4.2.3) and (4.2.4) are solved numerically for different values
of \( \dot{M} \). For a fixed value of \( \dot{M} \), the equations are solved for a range of
values of \( u \) and \( c \).

The accretion rate is found by comparing the mass flux with the
mass of the central object.

The accretion rate is given by

\[ \dot{M} = \frac{4\pi G M}{c^3} \left( \frac{u}{c} \right)^2 \]

(4.2.12)

The mass flux is constant, independent of \( r \).

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.13)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.14)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.15)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.16)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.17)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.18)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.19)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.20)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.21)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.22)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.23)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.24)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.25)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.26)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.27)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.28)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.29)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.30)

The density profile is given by

\[ \rho(r) = \frac{m}{V} \]

(4.2.31)

and the velocity profile is given by

\[ v(r) = \frac{c}{r} \]

(4.2.32)
to our desired boundary condition. Hence we shall restrict attention to the lower curve.

We must still decide which type of lower curve is reasonable—one that remains always below the bisectrix (case 1), or one that crosses the bisectrix (case 2). In case 1 the flow is subsonic at all radii, \( u < a \). But this requires a large back-pressure at all radii to retard the inflow—back pressure that can never be provided near the gravitational radius of a black hole. The gas must cross the gravitational

radius with the speed of light, as measured in the proper reference frame of an observer there; hence, the flow is surely supersonic near the gravitational radius.

Thus, the only acceptable solution \( a(u) \) is the lower curve of case 2 (Figure 4.2.2a). This curve begins at \( u = u_\infty = 0, a = a_\infty = 0 \); and it crosses from the subsonic region into the supersonic region at \( r = r_0 \). This transition to supersonic flow is crucial to the accretion process. It governs the accretion rate \( \dot{M}_0 \), in the same manner as the transition to supersonic flow in the throat of a rocket nozzle governs the rate of mass flow through the nozzle. In both cases one and only one mass flow rate is compatible with the required transition to supersonic flow.

Let us calculate the required accretion rate \( \dot{M}_0 \). We begin by calculating the

![Figure 4.2.1](image1.png)  
*Figure 4.2.1. The \( u, a \) "solution plane" for solving the coupled Bernoulli equation (4.2.3) and law of mass conservation (4.2.4).*

![Figure 4.2.2a](image2.png)  
*Figure 4.2.2a. Possible forms of adiabatic gas flow in a spherical, Newtonian gravitational field. Curves (a) [case 1 in text] correspond to flow that remains subsonic or supersonic everywhere. Curves (b) [case 2 in text] correspond to flow for which there is a sonic point, \( u = a \).*
flow velocity at the transition radius ("sonic radius") \( r_S \). To do this, we rewrite the Euler equation (4.2.2) in the form

\[
\frac{du}{dr} = \frac{a^2}{\rho} \frac{dp}{dr} - \frac{GM}{r^2}
\]

(4.2.5)

We differentiate the law of mass conservation (4.2.1) with respect to \( r \) and put \( du/dr \) from it into (4.2.5). As a result we obtain

\[
\frac{dp}{dr} \left( \frac{a^2 - u^2}{\rho} \right) = -\frac{GM}{r^2} + \frac{2u^2}{r} \label{eq:4.2.6}
\]

which shows that at the sonic point, where \( a = u \),

\[
\frac{2u^2}{r_S} = \frac{GM}{r_S^3} \quad \text{or} \quad a^2 = \frac{GM}{r_S} \label{eq:4.2.7}
\]

By combining this result with the Bernoulli equation (4.2.3) we obtain the speed of sound at the sonic point in terms of the speed of sound at infinity

\[
a_S = u_S = a_\infty \left( \frac{2}{5 - 3\Gamma} \right)^{1/2} \label{eq:4.2.8}
\]

and using equation (4.2.7) we obtain the radius at the sonic point

\[
r_S = \left( \frac{5 - 3\Gamma}{4} \right)^{1/2} \frac{GM}{a_\infty^3}. \label{eq:4.2.9}
\]

(Notice that the gravitational potential \( GM/r_S \) at the sonic point is of order \( a_\infty^2 \).)

Now it is straightforward to calculate the accretion rate from equation (4.2.4):

\[
\dot{M}_0 = 4\pi r_S^2 a_S^2 \rho_\infty \left( \frac{a_\infty}{a_\infty} \right)^{\frac{1}{2}} \frac{T_\infty}{r_\infty} \quad \text{or} \quad \dot{M}_0 = 4\pi \frac{r_S^2}{r_\infty} \rho_\infty \left( \frac{a_\infty}{a_\infty} \right)^{\frac{1}{2}} \frac{T_\infty}{r_\infty} \label{eq:4.2.10}
\]

\[
4\pi \frac{r_S^2}{r_\infty} \rho_\infty \left( \frac{a_\infty}{a_\infty} \right)^{\frac{1}{2}} \frac{T_\infty}{r_\infty} \quad \text{or} \quad \dot{M}_0 = 4\pi \frac{r_S^2}{r_\infty} \rho_\infty \left( \frac{a_\infty}{a_\infty} \right)^{\frac{1}{2}} \frac{T_\infty}{r_\infty} \label{eq:4.2.11}
\]

Here \( \alpha \) is a constant of order unity which depends on \( \Gamma \):

\[
\alpha = \frac{\pi}{4\Gamma^{3/2}} \left( \frac{2}{5 - 3\Gamma} \right)^{1/2} \left( \frac{\Gamma - 1}{\Gamma} \right) \approx 1.5 \quad \text{for} \quad \Gamma = 1, \label{eq:4.2.12}
\]

\[
1.2 \quad \text{for} \quad \Gamma = 1.4, \label{eq:4.2.13}
\]

\[
0.3 \quad \text{for} \quad \Gamma = 5/3; \label{eq:4.2.14}
\]

and we have used the equation of state \( p_\infty = (\rho_\infty m_p) k T_\infty \) corresponding to a mean mass per particle of \( m_p \).

This expression for the accretion rate is the main result of our calculation. Notice that \( \dot{M}_0 \) depends on \( \Gamma \) only very weakly. At a typical temperature of \( T_\infty \sim 10^8 \) K the interstellar gas will be partially ionized, so when it is compressed some energy will go into ionization. This means that the value of \( \Gamma \) outside and near the sonic point will be somewhat below 5/3. A value of \( \Gamma = 1.4 \) might be reasonable for use in equations (4.2.10) and (4.2.11).

It is now straightforward to derive from equations (4.2.3) and (4.2.4) the radial distributions of all quantities. The general picture is as follows. Outside the "radius of influence"

\[
r_I = 2GM/a_\infty^2 = r_S(a_S/a_\infty)^2 = 10r_S \quad \text{for} \quad \Gamma = 1.4. \label{eq:4.2.15}
\]

the pull of the hole hardly makes itself felt, so \( \rho \) and \( a \) are nearly equal to their values at "infinity", \( \rho = \rho_\infty, a = a_\infty \). At \( r \approx r_I \) the gas begins to fall with significant velocity, \( \rho \sim a \), and the density and sound velocity begin to rise. After the gas passes through the sonic point \( r_S \), it is in near-free-fall

\[
\rho \approx (2GM/r)^{1/2} a_\infty (r/r_S)^{1/2} \quad \text{at} \quad r < r_S; \label{eq:4.2.16}
\]

so the law of mass conservation requires the density to increase as

\[
\rho \approx \frac{M_0}{4\pi^2 r_S^2} \left( \frac{a_\infty}{a_\infty} \right)^{1/2} \left( \frac{r_\infty}{r} \right)^{1/2} \label{eq:4.2.17}
\]

\[
\rho_\infty \gg a_\infty \quad \text{if} \quad \Gamma = 1.4 \quad \text{near sonic point}. \label{eq:4.2.18}
\]

For adiabatic compression, temperature rises as \( T \sim \rho^{1/2} \); consequently, at \( r < r_S \)

\[
T \approx \frac{a_\infty^{3/2}}{4\pi} \left( \frac{r_\infty}{r} \right)^{1/2} T_\infty \approx 0.5 T_\infty \left( \frac{r_\infty}{r} \right)^{1/2} \label{eq:4.2.19}
\]

The above solution, (4.2.10)-(4.2.13), for adiabatic hydrodynamic accretion is due originally to Bondi (1952).

Rewrite the accretion rate (4.2.10) for hydrodynamic flow in a form similar to that (4.1.3) for noninteracting particles

\[
\dot{M}_0 \approx 2T_\infty (\dot{c}/\dot{a}) \approx \frac{1}{2} \left( \frac{M_0}{2M_\infty} \right) \left( \frac{a_\infty}{a_\infty} \right)^{1/2} \frac{T_\infty}{r_\infty} \left( \frac{r_\infty}{r} \right)^{1/2} \label{eq:4.2.20}
\]

\[
\dot{M}_0 \approx \left( \frac{1}{2} \frac{M_0}{2M_\infty} \right) \left( \frac{a_\infty}{a_\infty} \right)^{1/2} \frac{T_\infty}{r_\infty} \left( \frac{r_\infty}{r} \right)^{1/2} \label{eq:4.2.21}
\]

Direct comparison, and use of the approximate equality between speed of sound \( a_\infty \) and proton speeds \( v_\infty \), shows that the hydrodynamic accretion rate is larger by a factor \( (c/a_\infty)^2 \approx 10^8 \) than the accretion rate for independent particles. The physical reason for this is clear: gas is distinguished from independent particles by the frequent collisions of its atoms; these collisions limit the growth of tangential velocities during infall, but permit radial velocities to grow.

Other, useful forms for the hydrodynamic accretion rate are

\[
\frac{d(M_0/M_\infty)}{dt} \sim 10^{-6} \left( \frac{M_\infty}{M_0} \right)^{1/2} \left( \frac{M_0}{10^{18} \text{ g cm}^{-3}} \right)^{1/2} \left( \frac{a_\infty}{10 \text{ km sec}^{-1}} \right)^{3/2} \label{eq:4.2.22}
\]

\[
\dot{M}_0 \approx \left( \frac{1}{2} \frac{M_0}{2M_\infty} \right) \left( \frac{a_\infty}{a_\infty} \right)^{1/2} \frac{T_\infty}{r_\infty} \left( \frac{r_\infty}{r} \right)^{1/2} \label{eq:4.2.23}
\]

\[
\dot{M}_0 \approx \left( \frac{1}{2} \frac{M_0}{2M_\infty} \right) \left( \frac{a_\infty}{a_\infty} \right)^{1/2} \frac{T_\infty}{r_\infty} \left( \frac{r_\infty}{r} \right)^{1/2} \label{eq:4.2.24}
\]
4.3 Thermal Bremsstrahlung from the Accreting Gas

The above idealized case of spherical, adiabatic, hydrodynamic accretion can be complicated by various physical processes. Consider, first, the effects of energy loss due to thermal bremsstrahlung. To calculate the bremsstrahlung most easily, we shall assume that the energy loss has a negligible effect on the thermal energy density of the gas, and then we shall check that this assumption is valid.

When the effects of bremsstrahlung losses are neglected, then the temperature, density, and velocity distributions retain their adiabatic forms (4.2.13). One can readily check that for \( T_\infty \sim 10^8 \text{K} \) and \( \Gamma \approx 1.4 \) the gas remains nonrelativistic, \( kT < m_\rho c^2 \) (but just barely so) down to \( r \approx r_p \). Consequently, the rate at which one gram of gas at radius \( r \) radiates thermal bremsstrahlung is [cf. eq. (2.1.29)]

\[
\begin{align*}
\epsilon_{fr} & = \left(5 \times 10^{20}\ \text{ergs/g sec} \right) \left(\rho/\rho_\infty\right) \left(T_k/10^8\ \text{K} \right) \left(r/r_p\right)^{1/2} \\
& = \left(5 \times 10^{-3}\ \text{ergs/g sec} \right) \left(\rho_\infty/10^{24}\ \text{g cm}^{-3} \right) \left(T_\infty/10^8\ \text{K} \right) \left(r/r_p\right)^{1/2} \left(r/r_p\right)^{-(\Gamma - 1)/2}
\end{align*}
\]

(4.3.1)

For comparison, the rate at which radiative bremsstrahlung increases the energy of one gram of gas is [eq. (2.6.41)]

\[
\epsilon_{rad}\ \text{heating} = \frac{p d\rho}{\rho r^2} - \frac{p d\rho}{\rho r^2} \frac{u}{u} + \frac{3}{2} \frac{u}{r} = \frac{3}{2} \frac{u}{r} \left(\frac{\rho_\infty}{\rho_\infty}\right) \left(r/r_p\right)^{-3/2}
\]

(4.3.2)

Using expression (4.2.12) for \( r_p \), and the thermodynamic relations for a hydrogen gas

\[
a_m^2 = \left(\Gamma \rho_\infty \rho_m \right) \left(\rho \rho_m \right) \left(r/r_p\right)
\]

we can bring this heating rate into the form

\[
\begin{align*}
\epsilon_{rad}\ \text{heating} & = \left(3 \times 10^{4} \text{ergs/g sec} \right) \left(T_\infty/10^8\ \text{K} \right) \left(M_\infty/3 \right) \left(r/r_p\right)^{-3/2}
\end{align*}
\]

(4.3.4)

This heating rate greatly exceeds the free-free loss rate (4.3.1) at all radii. Hence, we are justified in our use of the adiabatic forms of \( T(r), \rho(r), \) and \( a(r) \). The total power radiated as thermal bremsstrahlung is

\[
L_{\text{ff}} \approx \int_{2}^{r_p} \epsilon_{fr} \rho_\infty m_r^2 \, dr
\]

\[
\sim \left(5 \times 10^{17}\ \text{ergs/g sec} \right) \left(M/10^{3}\ \text{M}_\odot\right)^{3} \left(\rho_\infty/10^{24}\ \text{g cm}^{-3} \right)^{2} \left(T_\infty/10^8\ \text{K} \right)^{3/2} \text{ for } \Gamma = 1.4.
\]

(4.3.5)

4.4 Influence of Magnetic Fields and Synchrotron Radiation

Up to now we have ignored the fact that the interstellar gas possesses a magnetic field. For interstellar temperatures of interest, \( T_\infty \sim 10^8 \text{K} \), the magnetic field will be frozen into the gas (cf. §2.5). The strength of the typical intergalactic field, \( B_r \sim 10^{-6}\ \text{G} \), is such that its energy density and pressure are not far below those of the gas

\[
B_r^2/8\pi \approx 4 \times 10^{-14}\ \text{ergs/cm}^3, \quad \rho_B \Pi \approx 3 \times 10^{24}\ kT_\infty \sim 2 \times 10^{-12}\ \text{ergs/cm}^3.
\]

(4.4.1)

At large radii the field may be small enough that one can ignore its influence on the accretion. In this case the standard hydrodynamical inflow will stretch each fluid element out radially (cross-sectional area of fluid element \( \pi r^2 \), hence diameter \( r \), volume \( \rho^{-1} \pi r^3 \)), hence radial diameter \( \Delta \rho /\rho \), and radial thickness \( d^{1/2} \).

This radial stretch and tangential compression must quickly convert the initial field in the fluid element into a nearly radial field with strength

\[
B \propto (\text{cross sectional area}) \sim r^{-1} \propto r^{-2}
\]

(conservation of flux). Hence, the magnetic energy in a given fluid element will rise as

\[
E_{\text{mag}} = \frac{B_r^2}{8\pi} \text{(volume of element)} \propto r^{-4} \propto r^{-7/2}.
\]

(4.4.2)

Not long after the gas crosses the sonic point, this magnetic energy will become so large as to exceed the thermal energy in the fluid element

\[
E_{\text{therm}} = \frac{3}{2} kT \text{ (mass of element)} \propto r^{-3/2}(\Gamma - 1).
\]

(4.4.3)

At this point magnetic pressures must come into play, and the flow will cease to obey the standard adiabatic hydrodynamic laws of §4.2.

The form of the modified magnetohydrodynamic flow is not understood at all well today. The best guesses and calculations to date are those of Schwartzman (1971). They suggest a turbulent flow, with reconnection of field lines between adjacent "cells" (cf. §2.9), and with a rough equipartition of the gravitational potential energy between magnetic fields, kinetic inflow of gas, turbulent energy, and thermal kinetic energy of gas particles:

\[
\frac{\text{gravitational energy}}{\text{per unit mass}} = \frac{GM}{r} \sim \frac{B_r^2}{8\pi r} \sim 4da^2 \sim 4d^2 \nu_{\text{turb}} \sim 3K_{\text{mag}}.
\]

(4.4.4)
The mass density corresponding to this configuration will determine by the size of the mass conservation, $4\pi r^2 \rho dr = M_0$. The magnetic field in this region is not strongly influenced by the flow, so we can use equation (4.2.15) for $M_0$ and $M$.

$\rho \sim \frac{M_0}{4\pi r^2} \left( \frac{V}{c} \right)^{-3/2}$ (4.4.3)

Accompanying this is the tendency for the magnetic field to remain strong enough to prevent field line reconnection. The field lines will be strongly influenced by the flow, so we can use equation (4.2.15) for $M_0$ and $M$.

$\rho \sim \frac{M_0}{4\pi r^2} \left( \frac{V}{c} \right)^{-3/2}$ (4.4.4)

and, because the gas turns out to be optically thin, the total luminosity is

$\gamma_{\text{total}} \sim \frac{6 \times 10^{-12} \text{ erg cm}^{-3}}{(10^4 \text{ K})^{3/2}} \frac{\rho}{\beta_c^2}$ (4.4.5)

For comparison, the net mass-energy being accreted is

$\dot{M}_\text{net} \sim (2 \times 10^{-12} \text{ erg cm}^{-3}) (\frac{\rho}{\beta_c^2}) (\frac{\dot{m}}{\dot{m}_\text{crit}})^{3/2}$ (4.4.6)

Since most of the radiation comes from $r = 2r_0$, where the thermal energy is $> 10^3 \text{ erg cm}^{-3}$, the effects of the energy losses on the temperature will not be significant. However, for $M < 2 \times 10^3 M_\text{sun}$, the effects of the energy losses on the radiation will be significant.

$\rho \sim \left( \frac{10^4 \text{ Hz}}{10^4 \text{ Hz}} \right)^{3/2} \frac{\beta_c^2}{\dot{m}_\text{crit}^2}$ (4.4.7)

Since the radiation from $r = 2r_0$ strongly dominates the synchrotron output, the composite spectrum of all the radiation will also have a broad maximum at the pressure on the synchrotron radius, and a power-law falloff. Notice that the radiation from the first two effects is governed by the
4.6. The Hydrodynamical Approximation

The above model for accretion is mainly based on two assumptions: (1) the gas in the accretion disk is optically thin, and (2) the gas pressure is much smaller than the magnetic pressure. These assumptions are made in order to simplify the calculations and to obtain a clear picture of the main processes occurring in the accretion disk. In reality, these assumptions are not always valid, and more complex models are necessary to describe the accretion process accurately.

4.7. The Magnetic Field

The magnetic field in the accretion disk is an important factor in the dynamical behavior of the disk. The magnetic field is generated by the accretion process itself and is also influenced by the rotation of the black hole. The magnetic field can affect the accretion flow, the emission of radiation, and the structure of the accretion disk.

4.8. The Black Hole

The black hole is the central object of the accretion disk and is responsible for the accretion of matter onto it. The black hole has a strong gravitational field, which can affect the motion of the accretion disk and the accretion flow. The black hole also has a strong magnetic field, which can further affect the accretion process.

4.9. The Accretion Disk

The accretion disk is a rotating disk of gas and dust that surrounds the black hole. The accretion disk is heated by the accretion process, and this heating is responsible for the emission of radiation. The accretion disk is also influenced by the magnetic field, which can affect the structure and dynamics of the disk.

4.10. The Schwarzschild Radius

The Schwarzschild radius is the radius of a sphere that is just large enough to prevent any escape of light from it. The Schwarzschild radius is a characteristic length scale for the black hole, and it is related to the mass of the black hole by the formula $r_s = 2GM/c^2$, where $G$ is the gravitational constant, $M$ is the mass of the black hole, and $c$ is the speed of light.

4.11. The Eddington Limit

The Eddington limit is the maximum rate at which a black hole can accrete matter without losing mass. The Eddington limit is given by the formula $L_{\text{Edd}} = 4\pi GM^2/c^3$, where $L$ is the luminosity, $G$ is the gravitational constant, $M$ is the mass of the black hole, and $c$ is the speed of light.
The scale of interest for determining the accretion rate is the sonic radius
\[ r_s = \frac{5}{4} \frac{3GM}{c^2} \left( \frac{M}{M_\odot} \right) \left( \frac{T_w}{10^5 K} \right)^{-1}, \]
(4.6.3)
at which point \( T \approx T_w, \rho \approx \rho_s, \) so
\[ \frac{\lambda_p}{r_s} \approx \rho \left( \frac{T_w}{10^5 K} \right)^3 \left( \frac{M}{M_\odot} \right) \left( \frac{1}{10^{-26} g \text{ cm}^{-3}} \right)^{-1} \]
(4.6.4)
Thus, if one ignores the magnetic field, then the gas will not quite behave like a fluid near the sonic point—and it may fail even more to be fluid-like at small radii; cf. §13.4 of ZN.

The interstellar magnetic field, of strength \( B_s \sim 10^{-6} G, \) can "save" the hydrodynamic approximation. It deflects protons away from straight-line motion in a distance of the order of the Larmor radius
\[ \lambda_L = \frac{m_p c \omega}{eB} \approx (1 \times 10^8 \text{ cm}) \left( \frac{B}{10^{-6} G} \right)^{-1} \left( \frac{T}{10^5 K} \right)^{1/2}, \]
(4.6.5)
and this is true even when the thermal pressure of the plasma exceeds the magnetic pressure so that the field gets dragged about by the gas. The Larmor radius is small compared to all macroscopic scales of interest (e.g. small compared to \( r_g \) when one uses the values \( B = B_s \) and \( T = T_w \) appropriate to \( r_g \)). Hence, the hydrodynamical approximation is fairly well justified.

4.7 Gas Flow Near the Horizon

Near the horizon, \( r \approx r_g, \) relativistic effects will modify the Newtonian picture of accretion. It is not difficult to solve for the relativistic flow using the relativistic Bernoulli equation (2.5.33) and the law of mass conservation (2.5.23). Such a solution is analogous to the Newtonian hydrodynamical solution studied in §4.2. However, such a solution is partially irrelevant because it ignores the role of the magnetic field; and such a solution is not needed because the intensity of the gravitational pull, as viewed in stationary frames near the horizon, is so great as to guarantee supersonic flow with qualitatively the same form as free-particle fall. Thus from a knowledge of geodesic orbits one can infer the main features of the flow.

Consider, first, spherical accretion onto a nonrotating hole (Schwarzschild gravitational field). A fluid element in the accreting gas, like a freely falling particle, reaches the horizon in finite proper time. The moment of crossing the horizon is in no way peculiar, as seen by the fluid element. The density does not reach infinity there; and the temperature does not reach infinity. In fact, the equation for radial geodesics in the Schwarzschild geometry
\[ ds^2 = -(1 - r/r_g)c^2 \ dt^2 + \left( 1 - r/r_g \right)^{-1} \ dr^2 + r^2 d\Omega^2, \]
(4.7.1)
has the first integral
\[ (dr/d\tau)^2 = 2GM/r \text{ (for fall from near rest at } r \gg r_g), \]
(4.7.2)
and because the law of rest-mass conservation \( \nabla \cdot (\rho\mathbf{v}) = 0 \) has the first integral
\[ 4\pi \sqrt{-g} \rho (dr/d\tau) = \dot{M}_0, \]
(4.7.3)
the density of rest mass must vary as
\[ \rho = \frac{\dot{M}_0}{4\pi r^2} \left( \frac{r}{2GM} \right)^{1/2} \approx (6 \times 10^{-12} g/\text{cm}^3) (r/r_g)^{3/2}. \]
(4.7.4)
This must be as true near and inside the horizon, \( r \approx r_g, \) as it is in the Newtonian realm \( r \gg r_g. \) Similarly, the temperature will retain its Newtonian form (4.4.2):
\[ T \approx (10^{12} K)/(r_g/r), \]
(4.7.5)
and the synchrotron emissivity will retain its Newtonian value (4.4.4). However, the radiation which reaches infinity will be sharply cut off as the gas element passes through the horizon
\[ \left( \text{fraction of emitted radiation that reaches } r = \text{ infinity from infalling gas at radius } r \right) = \frac{27}{64} \left( 1 - \frac{r_g}{r} \right)^3. \]
(4.7.6)
In previous sections we have taken rough account of this cutoff and of the accompanying redshift by retaining the Newtonian luminosity down to \( r = r_g, \) and then arbitrarily ignoring all radiation from \( r < r_g. \)

In the case of a rotating (Kerr) hole, near and inside the ergosphere the dragging of inertial frames will swing the infalling gas into orbital rotation about the hole. As the gas approaches the horizon, its angular velocity as seen from infinity must approach the angular velocity of the horizon,
\[ \Omega = \Omega_{\text{horizon}} = \frac{a}{r_g^2 + a^2} \quad \text{(geometrized units, } c = G = 1) \]
(4.7.7)
\[ = \frac{c^3}{2GM} \left( \frac{1}{\sec} \frac{M}{M_\odot} \right)^{-1} \quad \text{for "maximally rotating hole", } a = M. \]
Here \( a \) is the hole's angular momentum per unit mass. Any extra-luminous hot spot in the gas (e.g. a region where magnetic field lines are reconnecting; cf. §2.9) must orbit the hole with this angular velocity as it falls inward. The brightness of the hole as seen at Earth will be modulated with a period
\[ P \approx 4\pi G^2 (10^{-4} \text{ sec}) (M/M_\odot) \] as a result of the Doppler shift of the spot's light and the bending of its rays. (One factor of \( 2\pi/\Omega \) is the orbital period; the other factor of \( 2\pi/\Omega \) is the light travel time between the radius at which the spot is located at the beginning of one period and the radius to which it has fallen at the end of the period.) Thus, one might expect quasiperiodic fluctuations in the light from a black hole living alone in the interstellar medium.
4.8 Accretion onto a Moving Hole

Turn attention now from a hole at rest in the interstellar medium, to a hole that moves with a speed much larger than the sound speed, \( u_m \gg a_m \). Examine the resulting accretion in the rest frame of the hole. At radii \( r \gg 2GM/u_m^2 \), the kinetic energy per unit mass, \( \frac{1}{2}u_m^2 \), is far larger than the potential energy, \( GM/r \); so the pull of the hole has no effect on the gas flow. At \( r \sim 2GM/u_m^2 \), the pull of the hole becomes significant. Because the flow is supersonic the gas will respond to that pull in the same manner as would noninteracting particles; its pressure cannot have an influence. (One can see this, for example, in the Euler equation

\[
\text{const.} = \frac{1}{2}v^2 \frac{GM}{r} \left( \frac{a^2}{\Gamma - 1} \right) \approx \frac{1}{2}v^2 \frac{GM}{r}, \tag{4.8.1}
\]

where the speed-of-sound (enthalpy) term—which accounts for pressure effects—is negligible compared to the kinetic-energy term.) Thus, the flow lines will be identical to the trajectories of test particles; they will be hyperbola (Fig. 4.8.1a).

After passing around the hole, the trajectories of test particles intersect (particle collisions; dotted line of Figure 4.8.1a). In the gas-dynamic case such intersection of flow lines is prevented by rapidly mounting pressure. Consequently, a shock front must develop around the black hole (Fig. 4.8.1b).

Outside the shock front the flow lines will be hyperbolic test-particle trajectories. Behind the shock front, they will not. The shock will be located roughly where potential energy equals kinetic energy, so its characteristic size \( l_5 \) as shown in Figure 4.8.1b will be

\[
l_5 \sim GM/u_m^2. \tag{4.8.2}
\]

In the shock, the gas will lose most of its velocity perpendicular to the shock front ("strong shock" in terminology of §2.7; "strong" because \( u_m \gg a_m \) (speed of gas)/(speed of sound) \( \gg 1 \) on front side), but it will retain all of its velocity parallel to the front (denote this by \( v_y \)). For a given gas element, if the remaining kinetic energy behind the front greatly exceeds the potential energy, \( \frac{1}{2}v_y^2 \gg GM/r \), then the gas element will escape the pull of the hole. If \( \frac{1}{2}v_y^2 \ll GM/r \), then it cannot escape. The dividing line between escape and capture is a dotted line in Figure 4.8.1b; and the corresponding impact parameter is labelled \( b_{\text{capture}} \).

We can calculate \( b_{\text{capture}} \) in order of magnitude by idealizing the shock as confined to the thin dotted region of Figure 4.8.1a. (Hoyle and Littleton 1939).

The value of \( b_{\text{capture}} \) will correspond to an orbit for which

\[
\frac{1}{2}v_y^2 = GM/x \quad \text{at} \quad y = 0.
\]

More careful calculations give accretion rates which differ from this by multiplicative factors of \( \sim 0.5 \) to 1.0 (Bondi and Hoyle 1944).

Notice that the accretion rate (4.8.4) for the supersonic case, and the rate (4.2.10) for a hole at rest are identical except for the replacement of sound speed \( a_m \) by speed of flow \( u_m \), and except for a numerical coefficient of order unity. In the intermediate case of \( u_m \sim a_m \) one can use the hybrid formula

\[
b_{\text{capture}} = 2GM/u_m^2. \tag{4.8.3}
\]

Since the mass flux in the gas at large radii is \( \rho u_m a_m \), the accretion rate is

\[
\dot{M}_0 = (\pi b_{\text{capture}}^2 \rho u_m a_m) = 4\pi G^2 M^2 \rho u_m a_m, \tag{4.8.4}
\]
\[ \dot{M}_0 \approx \frac{4\pi G^2 M^2 \rho_{\text{in}}}{(a_{\text{in}}^2 + a_{\text{out}}^2)^{3/2}} \times \frac{(M/M_\odot)^2 (\rho_{\text{in}}/10^{-24} \text{ g cm}^{-3})}{(u_{\text{in}}/10 \text{ km sec}^{-1})^2 + (a_{\text{in}}/10 \text{ km sec}^{-1})^2}^{3/2}. \]  

\[(4.8.5)\]

[Salpeter (1964) gave the first application of these formulas to black holes; all previous work dealt with normal stars.]

It is worth noting that a significant fraction of the kinetic energy released in the shock

\[ \frac{dE}{dt} \approx \frac{1}{2} M \dot{M}_0 \simeq \left( \frac{10^{24} \text{ erg sec}^{-1}}{\dot{M}_0} \right)^2 \left( \frac{\rho_{\text{in}}}{10^{-24} \text{ g cm}^{-3}} \right) \left( \frac{u_{\text{in}}}{10 \text{ km sec}^{-1}} \right)^{-1/2} \]  

may go into excitation of ionized atoms and thence into light as the atoms decay, thus producing a shock front that glows (albeit weakly). See, e.g., Pikel'ner (1961), and § 10 of Kaplan (1966).

The motion of the gas behind the shock is practically unknown, particularly when one takes account of magnetic fields. Many different types of instabilities may occur, both here and in the case of a stationary black hole. For example at radii \( r < 3 \), where the inflow has become supersonic again, turbulence may create shock waves which convert kinetic energy into heat (Bisnovatyi-Kogan and Sunyaev 1971). A variety of plasma instabilities may also arise. And the reconnection of magnetic field lines will surely be important. One can only guess that the overall, time-averaged picture will resemble the equipartition model of Schwartzman (§ 4.4).

One factor that may be important near the gravitational radius, where most of the luminosity is produced, is angular momentum. Suppose that the accreting gas has nonzero angular momentum about the direction of motion of the hole (e.g. in direction in Fig. 4.8.1). If the angular momentum per unit mass, \( L \), exceeds \( r_{\text{gc}} \), then centrifugal forces will become important before the infalling gas reaches the horizon; the gas will be thrown into circulating orbits; and only after viscous stresses have transported away the excess angular momentum will the gas fall down the hole. (See §§ 5 for a situation similar to this.) The resulting viscous heating and lengthened time spent outside the hole, as well as the changed flow pattern, may affect the outpouring luminosity significantly. Moreover, when the hole moves from a region with angular momentum of gas in one direction to a region with angular momentum in another, the gas flow may become particularly violent for a while near the horizon; and a strong flare of luminosity may be produced (Salpeter 1964, Schwartzman 1971). Almost nothing is known today about these issues.

Is the specific angular momentum of the accreting gas likely to exceed \( r_{\text{gc}} \)?

Interstellar gas is accreted from a region of diameter

\[ 2L_{\text{capture}} = \frac{2M}{u_{\text{in}}^2} \left( \frac{3 \times 10^{14} \text{ cm}}{M} \right) \left( \frac{u_{\text{in}}}{10 \text{ km sec}^{-1}} \right)^{-2} \]  

(4.8.7)

(For \( u_{\text{in}} < a_{\text{in}} \), we must replace \( u_{\text{in}} \) by \( a_{\text{in}} \)). The interstellar gas is turbulent with the turbulent velocity \( u_{\text{turb}} \) on scales \( l \) given roughly by

\[ u_{\text{turb}} \approx \left( \frac{10^6 \text{ cm sec}^{-1}}{3 \times 10^{20} \text{ cm}} \right)^{1/2} \]  

(4.8.8)

where \( q \) is a coefficient believed to lie between 1/2 and 1 (Kaplan and Pikel'ner 1970), and the coefficient \( 10^6 \text{ cm sec}^{-1} \) is taken from astronomical observations. Let us take \( q = 0.75 \) as a best guess. Then the specific angular momentum associated with this turbulence, for a gas element of size \( l \), is

\[ \dot{L}_{\text{turb}} \approx \frac{2M}{u_{\text{in}}^2} \left( \frac{3 \times 10^{16} \text{ cm}}{3 \times 10^{20} \text{ cm}} \right)^{1.75} \]  

(4.8.9)

Consequently, the specific angular momentum of the gas that accretes from a region of size \( l = 2L_{\text{capture}} \) should be

\[ \frac{\dot{\Delta}_{\text{turb}}}{\dot{\Delta}_{\text{gc}}} \approx 1 \times \left( \frac{3 \times 10^{16} \text{ cm}}{3 \times 10^{20} \text{ cm}} \right)^{3/2} \]  

(4.8.10)

Evidently, the angular momentum may be important in some cases and may not be in others.

### 4.9 Optical Appearance of Hole: Summary

In summary, a black hole alone in the interstellar medium may be expected to emit \( \sim 10^{39} \) to \( 10^{45} \text{ ergs sec}^{-1} \) of synchrotron radiation, concentrated in the optical part of the spectrum (\( \sim 10^{14} \text{ Hz} \) to \( 10^{15} \text{ Hz} \)) (§ 4.4). The output may fluctuate on timescales of \( \sim 10^{-2} \) to \( 10^{-4} \) seconds due to instabilities, and to orbital motion of hot spots about the hole (§§ 4.4, 4.7, 4.8). There may also be strong flares when the hole moves out of one turbulent cell of interstellar gas into another. The time interval between flares would be about

\[ \Delta t_{\text{flares}} \approx 2L_{\text{capture}}/u_{\text{in}} \approx (10^{10} \text{ km sec}^{-1})^{-1} \]  

(4.9.1)

It will probably be difficult observationally to distinguish accreting, isolated black holes from accreting, old neutron stars without magnetic fields. The only distinguishing feature will be emission from the surface of the neutron star.
5.2 Accretion in Binary Systems: The General Picture

Consider a close binary system with one component a "normal" star and the other a "compact" star (black hole or neutron star or white dwarf). "Close" means that the separation a between the centers of mass is within a factor 2 or so of the radius R_N of the normal star.

\[ a \ll R_N \]

The accretion rate \( \dot{M} \), the luminosity \( L \), and the orbital period \( P \) of the binary are related through the equation:

\[ \dot{M} = \frac{4 \pi a^3 \rho v}{G M} \]

where \( \rho \) is the density of the gas and \( v \) is the orbital velocity. For a given mass function, the binary parameters are determined by the mass of the primary star and the orbital period.

\[ M_2 \sim 2 M_0 \times 10^{32} \mathrm{g}, \quad P \sim 1 \mathrm{yr} \]

Note: This is an excerpt from a scientific paper discussing accretion in binary star systems, particularly focusing on the dynamics of gas flow around compact stars like neutron stars or black holes. The text references specific equations and conditions for accretion processes, including the relationship between the mass of the stars and their orbital parameters.
Analyze the flow of gas from the normal star to its compact companion using a coordinate system that co-rotates with the binary system (noninertial frame). In the Newtonian realm (i.e., everywhere except near the surface of the compact star), the Euler equation (2.5.30) for the flowing gas reduces to

$$\frac{dv}{dt} - \mathbf{v} \cdot \nabla \Phi_{\text{gc}} - \Omega \times \mathbf{v} - \frac{1}{\rho_0} \nabla p + \frac{2}{\rho_0} \mathbf{v} \cdot (\nabla \eta).$$

(5.2.4)

Here $d/dt$ is the time derivative moving with a fluid element

$$d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla;$$

(5.2.5)

$v$ is the gas velocity relative to the rotating frame; $p$ and $\rho_0$ are pressure and rest-mass density; $\Phi_{\text{gc}}$ is the "gravitational-plus-centrifugal" potential

$$\Phi_{\text{gc}} = -\frac{M_N}{r-r_N} - \frac{M_C}{r-r_C} - \frac{1}{2}(\Omega \times r)^2;$$

(5.2.6)

and we have included a viscous shear stress, with $\eta$ the coefficient of dynamic viscosity and $\gamma$ the shear. In the Euler equation the term $-\nabla \Phi_{\text{gc}}$ gives rise to gravitational and centrifugal accelerations; and the term $-\Omega \times \mathbf{v}$ gives rise to Coriolis accelerations.

Notice that when pressure and viscous accelerations are unimportant (test-particle motion), the equation of motion (5.2.4) is identical to that for a particle with electric charge $q$ moving in an electric field with potential $\Phi_{\text{gc}}/q$, and a uniform magnetic field with strength $B = \Omega Q = (\Omega Q)_{\text{gc}}$. Conservation of energy in this case requires $\Phi_{\text{gc}} + \frac{1}{2} \gamma^2 = \text{const}$. When pressure forces are taken into account, energy conservation [Bernoulli equation (2.5.38)] is modified to read

$$\Phi_{\text{gc}} + \frac{1}{2} \gamma^2 + w = \text{constant along flow lines.}$$

(5.2.7)

When viscous stresses are taken into account, one cannot write such a simple conservation law.

To deduce the qualitative nature of the gas flow from the "normal" star to its companion, one must have a clear picture of the potential $\Phi_{\text{gc}}$. Figure 5.2.1 gives such a picture for the orbital plane. Of particular interest is the equipotential curve marked "Roche lobe". If the surface of the normal star is inside its Roche lobe, then the only way it can dump gas onto its compact companion is by means of a "stellar wind", which blows gas off the star supernormally in all directions with only that gas blown toward the companion being captured. But if the surface of the normal star fills its Roche lobe, then it can dump gas continuously through the "Lagrangian point" $L_1$ (Fig. 5.2.1) onto its companion. In this case the steady-state flow, as governed by the Euler equation (5.2.5) and by the law of mass conservation

$$\nabla \cdot (\rho_0 \mathbf{v}) = -\partial \rho_0 / \partial t = 0,$$

(5.2.8)

will have the qualitative form shown in Fig. 5.2.2.
Viscous stresses in the orbiting disk gradually remove angular momentum from each gas element, permitting it to gradually spiral inward toward the compact star. The angular momentum removed is transported by the viscous stresses from the inner parts of the disk to the outer parts, and is then picked up and carried away by passing gas (see above). At the same time, the shearing orbital motion of the gas in the inner parts of the disk, acting against the viscous stresses, produces heat ("frictional heating"). Much more heat is generated than the gas can store, so most of the heat gets radiated away from the top and bottom faces of the disk.

The total energy radiated by a unit mass of gas during its passage inward through the disk must equal, approximately, the gravitational binding energy of the unit mass when it reaches the inner edge of the disk, $E_{\text{bind}}$. For a white dwarf or neutron star the inner edge of the disk is near the star's surface, so

$$E_{\text{bind}} \approx \frac{1}{2}(M_{\text{c}}/R_{\text{e}})$$
$$\approx 10^{-4} \text{ for white dwarf}$$
$$\approx 0.05 \text{ for neutron star.}$$

For a black hole the inner edge of the disk is at the last stable circular orbit, which has

$$E_{\text{bind}} \approx 0.057 \text{ for nonrotating hole}$$
$$E_{\text{bind}} \approx 0.42 \text{ for maximally rotating hole.}$$

Thus, in order of magnitude the total luminosity of the disk must be

$$L \sim 10^{-4} \dot{M}_{\odot} \sim (10^{34} \text{ ergs/sec})(\frac{\dot{M}}{10^{-3} M_{\odot}/\text{yr}})$$

for white dwarf,

$$L \sim 0.1 \dot{M}_{\odot} \sim (10^{37} \text{ ergs/sec})(\frac{\dot{M}}{10^{-3} M_{\odot}/\text{yr}})$$

for neutron star or hole.

Here $\dot{M}_{\odot}$ is the accretion rate. In the case of a neutron star or white dwarf, the gas hits the star (or its magnetic field) with a kinetic energy roughly equal to the binding energy $E_{\text{bind}}$. In a steady state the energy liberated by this collision must all get radiated away, so the star itself will emit a luminosity of the same order as that emitted by the disk. In the case of a black hole, by contrast, the disk is the only source of radiation.

If the total luminosity approaches the "Eddington limit"

$$L_{\text{crit}} \approx (1 \times 10^{38} \text{ ergs/sec})(M/M_{\odot})$$

(5.2.11a)

\*\*Approximately" because the compact star itself can feed energy into the disk by means of viscous stresses, and that energy can subsequently be converted to heat and radiated away; see below.
off the upper face of the disk. An equal flux $F$ will come off the lower face.

The steady-state structure of the disk is governed by four conservation laws (conservation of mass, angular momentum, energy, and vertical momentum); by the nature of the viscosity—i.e., value of $\eta$—and by the law of radiative transfer from the inside of the disk to its surface.

Rest-mass conservation
Mass flux must flow across a cylinder of radius $r$ at a rate equal to the accretion rate

$$-2\pi \Sigma \psi = \dot{M}_0.$$  (5.2.17)

Angular-momentum conservation
The rate at which angular momentum is carried inward across radius $r$ by inflowing gas must equal the rate at which viscous stresses carry angular momentum out of radius $r$, plus the rate $\dot{J}_c$ at which angular momentum is deposited in the compact star

$$\dot{M}_0 (Mr^3)^{1/2} = 2\pi r^2 \tau v_r \dot{J}_c,$$

where $\tau$ is some factor. The specific angular momentum $I_c$ deposited into the compact star cannot exceed the Keplerian angular momentum at the inner edge of the disk $r_i$; hence

$$\dot{J}_c = \dot{M}_0 (Mr^3)^{1/2}$$

for some $|\psi| \ll 1$. Solving for the product of stress and disk thickness, we obtain

$$2\pi \tau \dot{J}_c = \frac{\dot{M}_0}{2\pi r^2} \left( (Mr)^{1/2} - \frac{2}{3} (Mr^3)^{1/2} \right).$$

$$\simeq \frac{\dot{M}_0 (Mr)^{1/2}}{2\pi r^2}$$

for $r \gg r_f$. (5.2.18)

Notice that in a steady state the product $2\pi \tau \dot{J}_c$ is determined uniquely by the accretion rate and the mass of the hole. If, at some radius, $2\pi \tau \dot{J}_c$ is (temporarily) smaller than (5.2.18), then the viscous stresses there will not be sufficient to handle the mass flow $\dot{M}_0$. As a result, mass will accumulate in the deviant region, increasing thereby the disk thickness $h$ and/or the stresses $\tau$; there, and returning the deviant region to a steady-state structure. For a quantitative analysis of this “approach to steady state” see Thorne (1973b).

Energy conservation
Heat is generated in the disk by viscosity at a rate, per unit volume, given by

$$\epsilon = 2\pi \tau \dot{J}_c = 4\pi \eta \psi^2 = -2\pi \psi \dot{J}_c.$$
Thus, the heat generated per unit area is
\[ 2\tau_{\text{r}} = \frac{2\hbar \epsilon}{4\pi r^2} \left( 1 - \beta \right) \frac{1}{r} \]
and the heat generated between radii \( r_1 \) and \( r_2 \) is
\[ \int_{r_1}^{r_2} 2\tau_{\text{r}} 2\pi r dr = 2\pi M_0 \left[ \frac{M}{r_1} - \frac{3}{2} \beta \left( \frac{r_1}{r_2} \right)^{1/2} \right] \left[ \frac{M}{r_2} - \frac{3}{2} \beta \left( \frac{r_2}{r_1} \right)^{1/2} \right] \]
\[ \approx \frac{3}{2} \beta M_0 \left( \frac{M}{r_1} - \frac{M}{r_2} \right) \quad \text{for} \quad r_2 \gg r_1 \gg r_1. \]

It is instructive to examine the origin of this heat energy by asking what net rate energy is being deposited in the region \( r_1 < r < r_2 \). Gravitational potential energy is released at a rate \( M_0 (M/r_1 - M/r_2) \); but only half of this energy can go into heat. The other half must go into orbital kinetic energy (virial theorem). Thus, the rate at which gravitational energy gets converted into heat is
\[ \frac{1}{2} M_0 (M/r_1 - M/r_2). \]
The viscous stresses transport outward not only angular momentum, but also energy. The rate at which energy is transported across radius \( r \) is
\[ \dot{E} = \Omega \dot{\Omega} = \Omega (2\pi r^2 \cdot 2\pi r^2) = \dot{M}_0 (M/r) \left( 1 - \beta \right) \frac{1}{r^{1/2}}. \]
Thus, energy is deposited by viscosity between radii \( r_1 \) and \( r_2 \) at a rate
\[ \dot{M}_0 \left( \frac{M}{r_1} - \frac{3}{2} \beta \left( \frac{r_1}{r_2} \right)^{1/2} \right) \left( \frac{M}{r_2} - \frac{3}{2} \beta \left( \frac{r_2}{r_1} \right)^{1/2} \right). \]
This energy deposition rate plus the deposition rate for gravitational energy is equal to the total heat rate. Notice that at \( r \gg r_1 \), gravity accounts directly for only one-third of the heat. The remaining two-thirds is transported into the heating region by viscous stresses. Much of the early literature on disk accretion, e.g., Lynden-Bell (1969), failed to take account of energy transport by viscous stresses, and therefore underestimated by a factor 3 the heating at \( r \gg r_1 \). If none of the heat were radiated away, the thermal energy of the disk would be 3/2 times its gravitational potential energy; and its temperature would thus be
\[ T \sim \frac{(M/r)M_0}{10^{13} K} r^{-1/2} \]
Such temperatures are absurdly high. Thermal bremsstrahlung and other radiative processes will never permit such temperatures to arise. Instead, they will quickly remove almost all of the heat that is generated. This means that the total flux \( 2F \) from the top and bottom faces of the disk must equal the heating rate per unit area, as calculated above (the heat flows out of the disk vertically rather than radially because the disk is so thin):
\[ F = \frac{3}{8\pi r^2} \left( 1 - \beta \right) \frac{1}{r^{1/2}}. \]
The total power radiated is thus
\[ \dot{L} = \int_r^\infty 2F \cdot 2\pi r dr = \left( \frac{1}{2} \beta \right) M_0 \frac{M}{r}. \]
Of this total power radiated, \( \frac{1}{2} M_0 M/r_1 \) comes from gravity, while \( (1 - \beta)M_0 M/r_1 \) comes from the rotational energy of the star. 

\section*{Spectrum}
A detailed treatment of the spectrum radiated will be given in § 5.10. Here we only note that, if the radiation is blackbody, then the surface temperature of the disk at radius \( r \) must be
\[ T_r = \frac{8K}{b} \left( \frac{M_0}{10^{10} M_0/yr} \right)^{1/4} \left( \frac{M}{M_0} \right)^{-1/2} \left( \frac{M}{r} \right)^{3/4} \left( 1 - \beta \right) \frac{1}{r^{1/2}}. \]
Because most of the radiation is emitted from \( r/M \sim 10 \) for a black hole or neutron star, and from \( r/M \sim 10^4 \) for a dwarf, and because the black-body spectrum peaks at a photon energy of
\[ h \nu_{\text{max}} \approx 4.4 \times 10^{-4} \text{eV} (T_{r}^{6} K), \]
the spectrum of the total radiation from the disk should peak at
\[ h \nu_{\text{max}} \approx (1 \text{keV}) \left( \frac{M_0}{10^{10} M_0/yr} \right)^{1/4} \left( \frac{M}{M_0} \right)^{-1/2} \left( \frac{M}{r} \right)^{3/4} \text{for neutron star or hole} \]
\[ h \nu_{\text{max}} \approx (0.01 \text{keV}) \left( \frac{M_0}{10^{10} M_0/yr} \right)^{1/4} \text{for white dwarf}. \]
Thus, for reasonable accretion rates the disk can emit strongly in the X-ray region (>1 to 10 keV); but the X-ray spectrum should fall fairly rapidly with increasing energy. It actually turns out that electron-scattering opacity imparts the emission of blackbody radiation, and causes the spectrum to peak at energies higher than that (5.2.23). (See § 5.10.) However, the above estimates are still accurate to within a factor ~10.
**Vertical pressure balance**

The thickness of the disk is governed by a balance between vertical pressure force and the tidal gravitational force of the compact star ("vertical momentum conservation"):

\[ \frac{dp}{dz} = \rho_0 \times \text{acceleration of gravity} = \rho_0 \frac{M_*}{r^3}. \]  

(5.2.24)

The approximate solution to this equation is

\[ h \simeq (\rho/\rho_0)^{1/2} (r^2/M) \Omega \simeq c_s/\Omega. \]  

(5.2.25)

Here \( h \) is the half-thickness of the disk, \( c_s \) is the speed of sound in the gas, and \( \Omega = (M/r^3)^{1/2} \) is the angular velocity of the gas.

**Sources of viscosity**

All of the above conclusions are based on conservation laws alone. To proceed further, one must make some assumption about the magnitude of the viscosity. The dominant sources of viscosity are probably chaotic magnetic fields, and turbulence in the gas flow.

For cases of interest the normal star (typically a B0 supergiant) could have a surface magnetic field of \( B_s \approx 100 \) gauss. Field lines will be dragged, by the flowing gas, off of the normal star and into the disk. The deposited field should be rather chaotic because there is no preferred direction for the field in the gas that flows off of the normal star. Turbulence, if present in the disk, will also make the field chaotic. Once the chaotic field has been deposited in the disk, the shear of the gas flow will magnify it at a rate (eq. (2.5.20))

\[ dB_s/dr = \alpha_B B_s \approx \Omega B_s \quad \text{and} \quad dB_t/dr = 0 \]  

(5.2.26)

corresponding to an increase of \( B_s \) by the amount \( B_s \) with every circuit around the compact star. This growth of field will be counterbalanced, at least in part, by reconnection of field lines at the interfaces between chaotic cells (see §2.9), and perhaps also by a bulging of field lines out of the disk, pinch-off of field lines, and escape of "magnetic bubbles." From a macroscopic viewpoint (scale large compared to chaotic cells) the effects of the field can be described by pressure and viscosity. The magnetic pressure will be of order \( B^2 \gamma \Omega \), where \( B \) is the mean field strength. Because shearing of the chaotic field will tend to string it out along the \( \phi \)-direction, its shear stress \( \tau_{\phi} \) will be somewhat (perhaps a factor 10?) smaller than its pressure

\[ \tau_{\phi}^{(mag)} \lesssim \rho_{mag} \frac{B^2}{\gamma \Omega}. \]  

(5.2.27)

The magnetic pressure cannot exceed the thermal pressure

\[ p_{mag}^{(mag)} \simeq \rho_{mag} \frac{c_s^2}{\gamma}. \]  

(5.2.28)

otherwise the field lines would bulge out of the disk, reconnect, and escape ("bubbles"). Thus, the magnetic viscous stresses will satisfy

\[ \tau_{\phi}^{(mag)} \lesssim \rho_{mag} \frac{c_s^2}{\gamma}. \]  

(5.2.29)

(Here \( c_s \) is the speed of sound, and \( p \) is the total pressure.)

The gas flow in the disk may well be turbulent. The coefficient of dynamic viscosity associated with turbulence is

\[ \eta \simeq \rho_0 l_{turb} \tau_{turb}. \]  

(5.2.30)

where \( l_{turb} \) is the speed of the turbulent motions relative to the mean rest frame of the gas, and \( \tau_{turb} \) is the characteristic size of the largest turbulent cells [see, e.g., §3 of Landau and Lifshitz (1959)]. If the turbulent speed ever exceeds the sound speed, then shocks develop and quickly convert the turbulent energy into heat. Thus, \( \tau_{turb} \lesssim c_s \). The turbulent scale is limited by the disk thickness, \( l_{turb} \lesssim h \). Consequently, the shear stress due to turbulence is bounded by

\[ \tau_{\phi}^{(turb)} \lesssim \eta \rho_0 \frac{\Omega}{\tau_{turb}} \lesssim \rho_0 \frac{c_s^2}{\gamma} \Omega \lesssim \rho_0 c_s^2 \lesssim p. \]  

(5.2.31)

Inequalities (5.2.29) and (5.2.31) on the magnetic and turbulent stresses are identical. They suggest that one lump one’s lack of knowledge about the true magnitude of the viscosity into a single parameter \( \alpha \) defined by

\[ \tau_{\phi} = \alpha p. \]  

(5.2.32)

Someday, perhaps ten years hence, when one understands the magnetic and turbulent viscosities better, one can insert into the formalism a reliable value of \( \alpha \). In the meantime one only knows that

\[ \alpha \lesssim 1. \]  

(5.2.33)

If \( \alpha \approx 1 \), the disk will have a rather mottled structure on scales of order \( h \); if \( \alpha \ll 1 \) it will be rather smooth on such scales.

**Radiative transport**

The heat generated by viscosity must be transported vertically to the surface of the disk before it can be radiated. The disk turns out to be optically thick (\( \tau_r + \tau_\alpha \gg 1 \) at \( z = 0 \)). Hence, one calculates the energy transport using the diffusion approximation [eq. (2.6.43) reduced to Newtonian form]

\[ \frac{d}{dz} \left( \frac{1}{2} b T_a^4 \right) = \kappa \rho_0 \frac{d}{dz} \]  

The approximate solution to this equation of transport is

\[ b T^4 \approx \xi \Sigma F. \]  

(5.2.34)
The dominant source of opacity is the outer parts of the disk, where the temperature is higher and electron scattering is the dominant source of opacity, as

\[
\epsilon \approx \frac{\kappa}{\kappa_{	ext{e}}/\kappa_{	ext{m}}} \approx 0.4 \times 10^4 \text{ cm}^{-2}
\]

(5.2.35a)

Throughout most of the disk, gas pressure dominates over radiation pressure, so

\[
\rho = p\rho_0 = \left(\frac{T}{T_0}\right)^{3/2}\rho_0
\]

(5.2.36a)

For a steady-state disk, the disk is governed by the law of mass conservation (\(\text{Eqs.}\ 5.2.18\)), the law of energy conservation (\(\text{Eqs.}\ 5.2.19\)), the law of angular momentum conservation (\(\text{Eqs.}\ 5.2.20\)), the law of momentum conservation (\(\text{Eqs.}\ 5.2.21\)), the law of radially symmetric pressure balance (\(\text{Eqs.}\ 5.2.22\)), the law of gravitational balance (\(\text{Eqs.}\ 5.2.23\)), the law of hydrostatic balance (\(\text{Eqs.}\ 5.2.24\)), the law of radially symmetric pressure balance (\(\text{Eqs.}\ 5.2.25\)), the law of gravitational balance (\(\text{Eqs.}\ 5.2.26\)), the law of hydrostatic balance (\(\text{Eqs.}\ 5.2.27\)), and the law of radiation pressure balance (\(\text{Eqs.}\ 5.2.28\)).

\[
m_{\text{max}} = \left(1 - 10^{17} \text{ Hz} \right)^{1/4} \left(\frac{R_0}{10^4 \text{pc}}\right)\frac{M_0}{10^6 \text{ M}_\odot}
\]

(5.3.1)

Thus, the radiation will be concentrated largely in the ultraviolet and optical region of the spectrum. If the disk cannot build up a blackbody spectrum (see §5.10 for details), then the radiation will be concentrated at somewhat higher frequencies (see §5.11).
build up around the disk. Such a cloud would absorb the optical and ultraviolet radiation and would reemit it in the far infrared. [See Lynden-Bell and Rees (1971) for details.] Hence, as seen from Earth the hole would be a strong source of infrared radiation \((L \sim 10^{42} \text{ to } 10^{44} \text{ ergs/sec depending on accretion rate})\).

Just such infrared sources are observed in the nucleus of our Galaxy \((L \sim 3 \times 10^{41} \text{ ergs/sec})\), in the nuclei of many other galaxies, and in quasars. Unfortunately, however, an accreting hole is not the only type of object that can generate such radiation. Any strong source of optical or ultraviolet radiation, surrounded by dust, will emit strongly in the infrared.

We shall see in § 5.12 that the disk around a supermassive hole may also emit a significant flux of radio waves, which will emit to Earth relatively freely.

### 5.4 Properties of the Kerr Metric Relevant to Accreting Disks

Before examining the details of the structure of an accreting disk (§ 5.9), we shall extend our equations of structure from Newtonian theory to general relativity. In this extension, we shall assume that the spacetime geometry outside the hole is that of Kerr, and we shall assume that the disk lies in the equatorial plane of the Kerr metric.

To treat the structure of such a disk, we shall need a number of properties of “direct” circular orbits in the equatorial plane of the Kerr metric (orbits that rotate in the same direction as the black hole). Most of our formulas are taken from Bardeen, Press and Teukolsky (1972), or Bardeen (1973)—or are readily derivable from results quoted there.

For simplicity in splitting formulas into Newtonian limits plus relativistic corrections, we shall introduce the following functions with value unity far from the hole:

\[
\mathcal{A} = 1 + \frac{a^2}{r^2} + 2 \frac{a}{r} \left( \frac{\dot{r}}{r} \right),
\]

\[
\mathcal{B} = 1 + \frac{a^2}{r^2},
\]

\[
\mathcal{C} = 1 - \frac{3M}{r} + \frac{2a}{r^2},
\]

\[
\mathcal{D} = 1 - \frac{2M}{r} + \frac{a^2}{r^2},
\]

\[
\mathcal{E} = 1 + \frac{2a^2}{r^2} - \frac{a^2}{r^2} + 3 \frac{a^2}{r},
\]

\[
\mathcal{F} = 1 - \frac{2a}{r^2} + \frac{a^2}{r^2},
\]

\[
\mathcal{G} = 1 - 2 \frac{a^2}{r^2} + \frac{a^2}{r},
\]

\[
\mathcal{H} = \exp \left[ -\frac{3}{2} \int \frac{\mathcal{F}^{3/2} - \mathcal{G}^{3/2}}{r} \right].
\]

\[
\mathcal{P} \equiv \frac{L - M_{\text{ini}}}{(M_{\text{ini}})^{1/2}} = \frac{\mathcal{H}^{1/2} - \mathcal{L}_{\text{ini}}}{(M_{\text{ini}})^{1/2}},
\]

\[
\mathcal{Q} \equiv \frac{L_{\text{ini}}}{(M_{\text{ini}})^{1/2}}.
\]

Here \(M\) and \(a\) are the mass and specific angular momentum of the black hole; \(r\) is radius; \(r_{\text{a}}\) and \(a_{\text{a}}\) are dimensionless measures of \(r\) and \(a\)

\[
r_{\text{a}} \equiv r/M, \quad a_{\text{a}} \equiv a/M;
\]

\(\mathcal{L}_{\text{ini}}\) is a constant defined below, and \(\mathcal{L}\) is a function of \(r\) defined below.

The properties of the direct circular orbits that we shall need are the following.

(i) Form of Kerr metric in and near equatorial plane \((\theta = \pi/2) \ll 1\):

\[
ds^2 = - \frac{r^2}{A} \mathcal{D} dt^2 + \frac{r}{A} \left( \mathcal{D} (\mathcal{D} - \omega dt)^2 + \frac{r^2}{\Delta} dr^2 + dz^2 \right) \]

\[
\Delta \equiv r^2 - 2Mr + a^2 = \mathcal{D}^2, \quad A \equiv r^4 + r^2 a^2 + 2Mr = r^4 \mathcal{F} \mathcal{G} \mathcal{E} \mathcal{D} \mathcal{Q} \]

\[
\omega = 2Mr/A (\mathcal{F} \mathcal{G} \mathcal{E} \mathcal{D} \mathcal{Q})^{-1} \]

We have replaced the usual angular coordinate \(\theta\) by \(z = r \cos \theta \approx r (\theta - \pi/2)\).

Note that the square root of the determinant of \(g_{\alpha\beta}\) is \((-g)^{1/2} = r\).

(ii) Angular velocity of orbit:

\[
\Omega = \frac{d\theta}{dt} = \frac{M^{1/2}}{r^{3/2} + aM^{1/2}} = \frac{1}{r^{3/2} \mathcal{B} \mathcal{F}^{1/2}} \]

(iii) Linear velocity of orbit relative to “locally nonrotating observer”:

\[
\mathcal{V}(\phi) = \frac{A}{r^{1/2} (\Omega - \omega)} = \mathcal{F}^{1/2} \mathcal{G}^{1/2} \mathcal{B}^{1/2} \]

(iv) “\(\gamma\)-factor” corresponding to this linear velocity:

\[
\gamma = (1 - \mathcal{V}(\phi)^{-1})^{-1/2} = \mathcal{F}^{1/2} \mathcal{G}^{1/2} \mathcal{B}^{1/2} \]

(v) Orthonormal frame attached to an orbiting particle with 4-velocity \(u\) (“orbiting frame”):

\[
\mathcal{E}_0 = \gamma \left( \frac{\mathcal{D} \dot{\mathcal{D}}}{A} + \frac{\dot{\mathcal{D}}}{\mathcal{D}} \right) = \mathcal{B} \left( \frac{\mathcal{D}^2}{A} + \frac{M^{1/2}}{r^{3/2}} + \frac{1}{\mathcal{D}} \right),
\]

\[
\mathcal{E}_\phi = \gamma \left( \frac{\mathcal{D} \dot{\mathcal{D}}}{A} + \gamma \mathcal{V}(\phi) \dot{\mathcal{D}} \right) = \mathcal{B} \left( \frac{\mathcal{D}^2}{A} + \omega \frac{\dot{\mathcal{D}}}{\mathcal{D}} + \omega \frac{\mathcal{G}^{1/2} \mathcal{B}^{1/2}}{\mathcal{D}^{1/2}} \right),
\]

\[
\mathcal{E}_r = \frac{\mathcal{D}^2}{A} \frac{\dot{\mathcal{D}}}{\mathcal{D}} = \mathcal{B} \left( \frac{\mathcal{D}^2}{A} + \omega \frac{\dot{\mathcal{D}}}{\mathcal{D}} \right),
\]

\[
\mathcal{E}_\theta = \frac{\Delta^{1/2}}{r} \frac{\dot{\mathcal{D}}}{\mathcal{D}} = \mathcal{B} \left( \frac{\mathcal{D}^2}{A} + \frac{\mathcal{G}^{1/2} \mathcal{B}^{1/2}}{\mathcal{D}^{1/2}} \right),
\]

\[
\mathcal{E}_\phi = \frac{\partial}{\partial \phi}.
\]
(vi) Corresponding orthonormal basis of one-forms
\[
\omega^b = \frac{\partial}{\partial \xi^1} dt - \frac{\partial}{\partial \xi^2} \left( \frac{M}{r} \right)^{1/2} r \, d\phi,
\]
\[
\omega^c = \frac{\partial}{\partial \xi^2} \left( \frac{M}{r} \right)^{1/2} \left[ r \, d\phi - \left( \frac{M}{r} \right)^{1/2} \frac{1}{\xi^1} \, dt \right],
\]
\[
\omega^\phi = \frac{\partial}{\partial \phi}.
\]  
(5.4.5b)

(vii) Shear of the congruence of circular, equatorial geodesics (congruence with 4-velocity \( u = u_0 \))
\[
\phi_{\theta\theta} = \phi_{\phi\phi} = \frac{1}{2} \frac{A}{r^2} \Omega^2 \theta = \frac{3}{4} \left( \frac{M}{r} \right)^{1/2} \frac{\partial}{\partial \theta},
\]
all other \( \phi_{\theta\phi} \) vanish.

(viii) Angular momentum per unit mass for circular orbit:
\[
\mathcal{L} = u_\phi = M^{1/2} r^{1/2} \phi^p \theta^{1/2}.
\]  
(5.4.7a)

(ix) Energy per unit mass for circular orbit:
\[
\mathcal{E} = |u_0| = \theta^{\theta^1/2}. 
\]  
(5.4.7b)

(x) Minimum radius for stable circular orbits ("marginally stable" orbits) \( r_{\text{ms}} \):
\[
r_{\text{ms}}^4 = 6Mr_{\text{ms}} + 8aM^{1/2}r_{\text{ms}} - 3a^2 = 0, \\
r_{\text{ms}}^4 M^3 \left( 3Z_3 - 3Z_1 + 2Z_2 \right)^{1/2},
\]
\[
Z_3 = (1 + a^2/M^2)^{1/3} \left[ (1 + q/M)^{1/3} + (1 - q/M)^{1/3} \right],
\]
\[
Z_2 = (3a^2/M^2 + Z_1)^{1/2}. 
\]  
(5.4.8a)

(xii) Angular momentum per unit mass for last stable circular orbit:
\[
\mathcal{L}_{\text{ms}} = \frac{2M}{3^{1/2} x} \left( 3x - 2a \right), \\
x = M^{1/2} r_{\text{ms}}.
\]  
(5.4.8b)

5.5 Relativistic Model for Disk: Underlying Assumptions

In order to avoid confusion, we shall lay more careful foundations for our relativistic analysis of disk structure than we did for the Newtonian analysis of § 5.2.

We shall split the calculation of disk structure into three parts: the radial structure (§§ 5.6), the vertical structure (§§ 5.7, 5.8) and the propagation of radiation from the disk’s surface to the observer.

In calculating the disk structure, we shall make the following assumptions and idealizations:

(i) The central plane of the disk coincides with the equatorial plane of the black hole.

(ii) The companion star in the binary system has negligible dynamical influence on the disk structure (assumption valid in inner part of disk; not at outer edges).

(iii) The disk is thin, i.e., its proper thickness \( \delta \), as measured in the orbiting frame of the disk’s matter (same as proper thickness measured by any other observer who moves in the equatorial plane), satisfies
\[
\delta(r) \ll r. 
\]  
(5.5.1)

(iv) The disk is in a quasi-steady state. This assumption can be made precise in terms of an averaging process. Pick a particular height \( z(z \leq \delta < r) \) above the central plane of the disk, and pick a particular quantity \( \Psi \) (e.g., \( \Psi = \text{density or temperature or 4-velocity of gas} \) to be measured at height \( z \). Average \( \Psi \) over all \( \phi \) (Lie dragging it along \( \partial/\partial \phi \) while averaging, if it is a vector or tensor); average over a proper radial distance of order \( 2h \); and average over the time interval required for gas to move inward a distance \( 2h \):
\[
(\Psi) = \frac{1}{2\pi r \Delta \theta} \int_{-\pi/2}^{\pi/2} \int_{r-h}^{r+h} \Psi \, d\phi \, dt 
\]  
(5.52)

where \( \Delta r \) will be defined below. The assumption of a quasi-steady state means that (\( \Psi \) is time-independent, i.e.,
\[
\partial(\Psi)/\partial t = 0 \quad \text{if} \quad (\Psi) \text{ is a scalar,}
\]
\[
\mathcal{L}_{\partial r}(\Psi) = 0 \quad \text{if} \quad (\Psi) \text{ is a vector or tensor,}
\]  
(5.5.3)

where \( \mathcal{L}_{\partial r} \) is the Lie derivative along the Killing vector \( \partial/\partial t \). The disk structure may be very far from steady state on scales \( \lesssim h \); for example, it may have turbulence, flux reconnection, and flares on such scales. Such local violence will not invalidate the analysis that follows.

(v) When viewed "macroscopically" (turbulence removed by the above averaging process), the gas of the disk moves (very nearly) in direct, circular, geodesic orbits—i.e., with 4-velocity \( \omega \equiv u_0 \). Superimposed on this orbital motion is a very small radial flow, produced by viscous stresses, and an even smaller flow in the vertical direction, as required by the variation of disk thickness with radius:
\[
(\omega) = \frac{\partial}{\partial \xi^3} \frac{\partial}{\partial \xi^1} \left[ 1 - (\xi^2)^2 - (\xi^3)^2 \right]^{1/2} + v^r \theta^t + v^\phi \phi^t.
\]  
(5.5.4)
\[ |v^2(r, z)| \ll |v^2(r, z)| \ll \gamma^2 \approx (M/r)^{1/2}. \tag{5.5.5} \]

By assuming that the orbital motion of the gas is very nearly geodesic, we are automatically requiring that the gravitational pull of the hole dominates over radial pressure gradients and over shear stresses. In order of magnitude the gravitational pull of the hole ("gravitational acceleration") is the gradient of the gravitational binding energy, \((1 - \frac{\nu}{c})\), and the acceleration due to pressure gradients and shears is \(\sim (T_{\theta \theta}/\rho_0) \sim (T_{\phi \phi}/\rho_0)\). Thus, we are automatically demanding that the stress tensor in the fluid’s rest frame (\(\approx\) orbiting orthonormal frame) satisfy
\[
\frac{T_{\theta \theta}(r, z)}{\rho_0(r, z)} \ll 1 - \frac{\nu}{c} \approx \frac{1}{2} \frac{M}{r}. \tag{5.5.6} \]

(Here \(\approx\) means "equals in the Newtonian limit").

Because the specific internal energy \(\Pi\) is approximately equal to \(T_{\theta \theta}/\rho_0\), the assumption of nearly geodesic orbits implies the condition of "negligible specific heat":
\[
\Pi(r, z) \ll 1 - \frac{\nu}{c} \approx \frac{1}{2} \frac{M}{r}. \tag{5.5.7} \]

In words—the density of internal energy, \(\rho_0\Pi\) (including thermal energy of gas, energy of turbulent motions, and magnetic-field energy)—is much smaller than the density of gravitational binding energy \(\rho_0(1 - \nu/c)\).

Notice that conditions (5.5.6) and (5.5.7) imply
\[
\Pi \ll 1, \quad T_{\theta \theta}/\rho_0 \ll 1. \tag{5.5.8} \]

at all \(r\) and \(z\)—even near the black hole. This means that, although general relativistic effects (spacetime curvature; deviations from Newtonian gravity) are very important near the hole, one can everywhere ignore special relativistic corrections to the local thermodynamic, hydrodynamic, and radiative properties of the gas. Contrast this situation with the interior of a neutron star and the early stages of the Universe, where not only is gravity relativistic but so is the matter \((\Pi \gtrsim 1, \rho(\gamma_0 \gtrsim 1, T_{\phi \phi}/\rho_0 \gtrsim 1))\).

Turn now from the underlying assumptions of the model to the reference frames to be used in the calculations. Three reference frames will be needed: (i) The Boyer-Lindquist coordinate frame \((t, r, z, \phi)\) inside and near disk; \((t, r, \theta, \phi)\) when \(|\theta - \pi/2| \neq 1\); see eqs. (5.4.2a). (ii) The orbiting orthonormal frame \([e_\theta, e_\phi, e_\nu, e_\zeta]\); eqs. (5.4.5). (iii) The mean local rest-frame of the gas
\[
e_\theta \equiv (u) \quad \text{as given by eq. (5.5.4)},
\]
\[
e_\phi \equiv e_\phi,
\]
\[
e_\nu \equiv (e_\nu + u^\nu e_\nu)/|1 + u^\nu e_\nu|,
\]
\[
e_\zeta \equiv (e_\zeta + u^\zeta e_\zeta - u^\nu e_\nu)/|1 + u^\nu e_\nu - u^\zeta e_\zeta|.
\]

The mean local rest frame is nearly identical to the orbiting orthonormal frame.

### 5.6 Equations of Radial Structure

The relativistic equations of radial structure for our disk will be expressed in terms of the following parameters: (i) The steady-state accretion rate \(\dot{M}_0\); (ii) The surface density of the disk
\[
\Sigma \equiv \int^0_{-h} \langle \rho_0 \rangle \, dz, \tag{5.6.1a} \]
where \(\rho_0\) is the density of rest mass as measured in the rest frame of the gas.

(iii) The integrated shear stress
\[
W \equiv \int^0_{-h} (T_{\theta \phi}) \, dz, \tag{5.6.1b} \]
where \(T_{\theta \phi}\) is the component of the stress-energy tensor on the orbiting orthonormal basis vectors \(e_\theta\) and \(e_\phi\).

(iv) The mass-averaged radial velocity of the gas
\[
v^\zeta \equiv \langle 1/\Sigma \rangle \int^0_{-h} \langle v^\zeta \rho_0 \rangle \, dz. \tag{5.6.1c} \]

(v) The flux of radiant energy off the upper face of the disk (equal also to flux off lower face)
\[
F \equiv \langle (T^0_\theta (z = h)) \rangle = \langle -T^0_\phi (z = -h) \rangle. \tag{5.6.1d} \]

(vi) The mass \(M\) and specific angular momentum \(a\) of the hole. (vii) The radial coordinate \(r\).

The laws governing the radial structure are conservation of rest mass, conservation of angular momentum, and conservation of energy.

**Conservation of rest mass**

The amount of rest mass that flows inward across of cylinder \(\pi \Delta r\) during coordinate time \(\Delta t\), when averaged by the method of equation (5.5.2), must equal \(\dot{M}_0 \Delta t\) (conservation of rest mass). The mass transferred can be written
as the flux integral

\[ \tilde{M}_0 \Delta t = \int \rho \, d^3 \Sigma = \int_{0}^{\Delta r} \left( \rho \, \hat{y} \right) \, dz, \]

where \( \Sigma \) is the 3-surface \( 0 \leq \varphi \leq \pi, -h < z < h, 0 \leq \tau < \Delta \tau \). Rewritten in terms of the mass-averaged radial velocity \( [eq. (5.6.1)] \), this equation becomes

\[ \tilde{M}_0 = -2\pi \varphi \Sigma \hat{y} \left| \frac{\hat{r}}{\dot{r}} \right| \text{ (constant, independent of } \tau \text{ and } t \text{) } \]  

(5.6.2)

**Conservation of angular momentum**

The law of conservation of angular momentum can be written in the form

\[ \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = \mathbf{T} \cdot \hat{\mathbf{n}}, \]

(5.6.3)

where \( \hat{\mathbf{n}} \) is the Killing vector associated with rotation about the symmetry axis. Without any loss of generality, we can write the stress-energy tensor \( \mathbf{T} \) in the form

\[ \mathbf{T} = \rho(1 + \Pi) \mathbf{u} \mathbf{u} + \mathbf{t} \cdot \mathbf{u} \mathbf{q} + \mathbf{q} \cdot \mathbf{u} \mathbf{q}, \]

(5.6.4a)

where \( \Pi \) is specific internal energy, \( \mathbf{t} \) is the stress tensor as measured in the local rest frame of the baryons, \( \mathbf{q} \) is the energy flux relative to the local rest frame (see \$2.5\$), and

\[ \mathbf{u} \cdot \mathbf{t} = \mathbf{u} \cdot \mathbf{q} = 0. \]

(5.6.4b)

The corresponding density of angular momentum is

\[ J^a = \rho a^2 \mathbf{u}^a + \mathbf{t}^a + \mathbf{q}^a \mathbf{u}^a, \]

where we have dropped the angular momentum \( \rho a^2 \mathbf{u}^a \mathbf{u}^a \) associated with the internal energy because \( \Pi \ll 1 \). The law of angular-momentum conservation thus reads

\[ \nabla \cdot \mathbf{J} = 0 = \rho \frac{d a}{d t} + \mathbf{u} \cdot \nabla \mathbf{q} + \mathbf{q} \cdot \nabla \mathbf{u} + \nabla \cdot (\rho \mathbf{a} \mathbf{u}). \]

[Here we have used the law of rest-mass conservation, \( \nabla \cdot (\rho \mathbf{u}) = 0 \).] Average this equation over \( \tau, \varphi, r \), and integrate over \( z \). The result is

\[ \int_{-h}^{h} \frac{d a}{d t} \frac{d z}{d t} + \int_{r}^{\infty} \left( q^a \mathbf{u}^a \right) \right] \left[ \nabla \cdot (\rho \mathbf{a} \mathbf{u}) \right] + 2 (\mathbf{u} \cdot \mathbf{F}) \right] F = 0. \]

[Here we have invoked the thinness of the disk to infer that \( \mathbf{q} \) is in the vertical, \( \mathbf{e}_z \) direction, we have used stationarity, axial symmetry, and reflection symmetry about \( z = 0 \) to discard several terms; and we have used the relation \( (q^a \mathbf{u}^a) = (q^a \mathbf{u}^a) = \left( -\hat{a}^a \right) \right) \right] \right] \left[ \nabla \cdot (\rho \mathbf{a} \mathbf{u}) \right] + 2 (\mathbf{u} \cdot \mathbf{F}) \right] F = 0. \]

Inserting this and the relation

\[ (d u^a / d t) = (u^a u^b) = \hat{a}^a \hat{a}^b \hat{a}^c \]

into the conservation law, obtain

\[ \hat{a}^a \hat{a}^b \hat{a}^c + r^{-1}(r^2 \mathbf{a} \cdot \mathbf{e}^{-1/2} \mathbf{W}) \cdot r + 2\hat{L} F = 0. \]

(5.6.5)

The first term is the rate of increase of angular momentum in the gas; the second is the rate at which shear stresses carry off angular momentum; the third is the rate at which photons carry off angular momentum. When combined with the law of mass conservation (5.6.2), this law of angular momentum conservation takes on the simpler form

\[ (\hat{a}^a \hat{a}^b \hat{a}^c + r^{-1}(r^2 \mathbf{a} \cdot \mathbf{e}^{-1/2} \mathbf{W}) \cdot r + 2\hat{L} F = 0. \]

(5.6.6)

**Conservation of energy**

Turn now to the law of energy conservation

\[ \mathbf{u} \cdot (\nabla \cdot \mathbf{T}) = 0. \]

(5.6.7)

Rewrite this law in the form

\[ \nabla \cdot (\mathbf{u} \cdot \mathbf{T} - u_{00} T_{00}) = 0. \]

(5.6.8)

Use the general stress-energy tensor (5.6.4) to write

\[ \mathbf{u} \cdot \mathbf{T} = -\rho(1 + \Pi) \mathbf{u} - \mathbf{q}; \]

and combine this with the law of rest-mass conservation \( \nabla \cdot (\rho \mathbf{u}) = 0 \) to obtain

\[ \nabla \cdot (\mathbf{u} \cdot \mathbf{T} - u_{00} T_{00}) = 0. \]

(5.6.9)

Decompose \( u_{00} T_{00} \) into its irreducible tensorial parts

\[ u_{00} T_{00} = \omega \phi \phi + a_{00} + \frac{1}{2} \theta \theta \]

[eq. (2.17)] \right], and contract it into the stress-energy tensor (5.6.4a) to obtain

\[ u_{00} T_{00} = a_{00} \phi \phi + \frac{1}{2} \theta \theta - \mathbf{a} \cdot \mathbf{q}. \]

(5.6.10)

Then use relations (5.6.9) and (5.6.10) to rewrite the law of energy conservation (5.6.8) in the form

\[ \rho \frac{d \Pi}{d t} = \mathbf{u} \cdot \mathbf{q} = -a_{00} \phi \phi - \frac{1}{2} \theta \theta - \mathbf{a} \cdot \mathbf{q}. \]

(5.6.11)

The various terms in this equation have simple interpretations. The left-hand side represents the rate at which the energy of the baryons is distributed locally in the gas: \( \rho \frac{d \Pi}{d t} \) is the energy going into internal forms; \( \mathbf{u} \cdot \mathbf{q} \) is the energy being transported out of the region of generation. The right-hand side of this equation represents the rate at which energy is generated: \( a_{00} \phi \phi \) is the energy being generated by "frictional" (viscous) heating; \( -\frac{1}{2} \theta \theta \) is the energy being fed in by compression. The remain-
The corresponding value of the flux, as obtained from the law of energy conservation (5.6.12), is

$$F = \frac{3M_0}{8\pi^{2/3}} \frac{\mathbf{B}}{r^{1/3}}.$$  \hspace{1cm} (5.6.14b)

The final equations of radial structure are these two equations for \(W\) and \(F\), plus the law of rest-mass conservation (5.6.2):

$$M_0 = -2\pi r \Sigma \dot{v}^{2/3}.$$  \hspace{1cm} (5.6.14c)

These are the relativistic, black-hole versions of the Newtonian equations of radial structure (5.2.17), (5.2.18), and (5.2.19).

The equations of radial structure can be viewed as three equations linking the four radial functions \(\Sigma\) (surface density), \(W\) (integrated stress), \(\dot{v}\) (mass-averaged radial velocity), and \(F\) (radial flux). Note that these equations determine \(W\) and \(F\) explicitly; but they determine only the product \(\Sigma \dot{v}^{2/3}\) not the individual functions \(\Sigma\) and \(\dot{v}^{2/3}\). To calculate \(\Sigma\) and \(\dot{v}^{2/3}\) individually—and to calculate other features of the disk such as thickness \(h\), internal temperature \(T\), etc.—one must build a model for the vertical structure.

### 5.7 Equations of Vertical Structure

The equations of radial structure (5.6.14) are based on conservation laws, without any reference to the detailed properties of the gas in the disk (no reference to equation of state, or to nature of viscous stresses, or to nature of opacity, or to nature of turbulence and magnetic fields). By contrast, the equations of vertical structure, discussed below, require explicit assumptions about the properties of the gas. Hence, almost all of the uncertainties and complications of the model are lumped into the vertical structure. Ten years hence one will have a much improved theory of the vertical structure, whereas the equations of (averaged, steady-state) radial structure will presumably be unchanged.

Inside the disk the characteristic scale on which the vertical structure changes is \(h\), while the characteristic scale for changes in the radial structure is \(r \gg h\). Consequently, with good accuracy an observer inside the disk can regard the local variables (temperature, density, etc.) as functions of height, \(z\), only. To analyze the local vertical structure most conveniently, one performs calculations in the local orbiting orthonormal frame \(e_2, e_3, e_4, e_5\), which is located in the central plane of the disk (\(z = 0\)). Aside from rotation about the \(e_2\) axis—which produces Coriolis and centrifugal forces—this frame is (locally) inertial. The Coriolis and centrifugal forces have no influence on the vertical structure; hence we can ignore them and throughout this section can regard the orbiting frame as inertial.

The laws of physics in the orbiting inertial frame at any given \(t, r, \varphi\) are those
of special relativity (equivalence principle). And because \( \Pi \sim \gamma \), we have those special relativistic laws take on their standard Newtonian forms. At least they do so if one uses the correct Kerr-metric value for the tidal gravitational acceleration which compresses the disk into its "pancake" shape:

\[
\text{("acceleration of gravity") } \equiv g = R_{\theta \phi}^{2b} \tag{5.7.1}
\]

(cf. eq. 5.2.24). An explicit expression for \( R_{\theta \phi}^{2b} \) can be obtained by a transformation of the "LNRF" components, \( R_{\theta \phi}^{(in)} \), of the Riemann tensor as given in Bardeen, Press, and Teukolsky (1972):

\[
R_{\theta \phi}^{2b} = \frac{M}{\rho_0^2 \gamma} \left[ \left( \left( \frac{\rho_0^2}{\rho_0^2 + \rho_0^2} \right) + 2\Delta a^2 \right)^{1/2} \right] = \frac{M}{\rho_0^2 \gamma} \left[ \left( \left( \frac{\rho_0^2}{\rho_0^2 + \rho_0^2} \right) + 2\Delta a^2 \right)^{1/2} \right] = \frac{M}{\rho_0^2 \gamma} \left[ \left( \left( \frac{\rho_0^2}{\rho_0^2 + \rho_0^2} \right) + 2\Delta a^2 \right)^{1/2} \right]. \tag{5.7.2}
\]

Thus, a Newtonian astrophysicist can build a relativistically correct vertical structure without knowing any relativity at all. He need merely follow standard Newtonian procedures and theory; but he must use expression (5.7.1) for the vertical acceleration of gravity, rather than the Newtonian formula \( g = (M/r^4)z \).

We shall describe the vertical structure in terms of the following functions of height—which we tacitly assume have all been averaged over \( \tau, \tau, \theta, \psi \) by the methods of equation (5.5.2):

\[
\begin{align*}
\rho_0(r, z) &= \text{density of rest mass}; \\
p(r, z) &= \text{vertical pressure}, \ T_{\theta \phi} \\
t_\phi(r, z) &= \text{shear stress}; \\
T(r, z) &= \text{temperature}; \\
q^\phi(r, z) &= \text{flux of energy}; \\
\bar{k}(r, z) &= \text{Rosseland mean opacity}.
\end{align*}
\tag{5.7.3}
\]

These 6 vertical structure functions are governed by the following 6 "equations of vertical structure":

Vertical pressure balance

\[
\frac{dp}{dz} = \rho_0 \times (\text{"acceleration of gravity")} = \rho_0 R_{\theta \phi}^{2b} z
\tag{5.7.4a}
\]

Sources of viscosity

\[
t_\phi \times \text{(some explicit expression which depends on the explicit assumptions of the model)} \tag{5.7.4b}
\]

Energy generation (cf. eq. (5.6.11))

\[
\frac{d\epsilon}{dz} = -2\epsilon \chi \left( \epsilon \right) \frac{T_{\theta \phi}}{\epsilon} = \left( M/r^2 \right)^{1/2} \epsilon \phi \epsilon^{-1} \gamma. \tag{5.7.4c}
\]

Energy transport

\[
q^\phi \times \text{(some explicit expression which depends on the nature of the transport assumed by the model)} \tag{5.7.4d}
\]

The energy transport may be by radiative diffusion in some models, but by turbulent gas motions in others. All models to date have assumed radiative transfer with large optical depth, \( \tau_{\alpha} \gg 1 \) (see §2.6). In this case

\[
q^\phi = \frac{1}{\rho_0} \frac{d}{dz} \left( \epsilon \phi \epsilon^{-1} \gamma \right). \tag{5.7.4d}
\]

Equation of state for vertical pressure

\[
p = \epsilon \phi \epsilon^{-1} \gamma \times \text{(some explicit expression which depends on the explicit assumptions of the model)} \tag{5.7.4e}
\]

Possible contributors to vertical pressure are thermal gas pressure, radiation pressure, magnetic pressure, and turbulent pressure. Most models to date have ignored magnetic pressure and turbulent pressure, and have therefore taken

\[
p = \rho_0 (T/m_{\text{mm}} m_p) T^4 = \rho_0 (T/m_{\text{mm}} m_p) T^4 \tag{5.7.4e}
\]

where \( m_{\text{mm}} \) is the mean molecular weight (0.5 for an ionized hydrogen gas).

Equation for opacity

\[
\bar{k} = \text{(some explicit expression which depends on the explicit assumptions of the model)} \tag{5.7.4f}
\]

Of these 6 equations of vertical structure, 3 are differential equations. They must be subjected to 3 boundary conditions. As in the theory of stellar structure, the boundary conditions at the surface, \( z = h \), must be a join to an atmosphere in which the diffusion approximation for radiative transfer (eq. 5.7.4d) is abandoned. Alternatively, one can use "zero-order boundary conditions" which ignore the atmosphere—and then "tack" an atmosphere onto the model afterwards. The zero-order version of the boundary conditions is as follows: (i) define the surface of the disk, \( z = h \), to be that point at which the density goes to zero

\[
\rho_0 = 0 \text{ at } z = h; \tag{5.7.5a}
\]

then (ii) temperature must also go to zero at the surface

\[
T = 0 \text{ at } z = h; \tag{5.7.5b}
\]
(iii) the flux must vanish on the central plane of the disk
\[ q^2 = 0 \quad \text{at} \quad z = 0; \]  
\( q^2 = 0 \quad \text{at} \quad z = 0; \quad (5.7.5c) \)

(iv) the integrated stress must equal the expression (5.6.14a) calculated from the theory of radial structure
\[ 2 \int_0^b \rho_0 \, dz = W. \]  
\( 2 \int_0^b \rho_0 \, dz = W. \quad (5.7.5d) \)

The equations of vertical structure (5.7.4) and these boundary conditions automatically guarantee that 3 of the equations of radial structure—(5.6.14a, b)—are satisfied. The vertical structure also provides one with a value for
\[ \Sigma = 2 \int_0^p \rho_0 \, dz, \]  
which one can insert into the third radial equation (5.6.14c) to obtain the mean radial velocity \( \dot{v}^r \). Then the entire structure, both radial and vertical, is known.

5.8 Approximate Version of Vertical Structure
Because of the great current uncertainties about the roles and forms of turbulence and magnetic fields, there is no justification in 1972 for building sophisticated models of vertical structure. Therefore, let us solve for the vertical structure in the same very approximate manner as we used in the Newtonian treatment of §5.2. In particular, let us replace the vertical functions \( \rho_0, \rho, \dot{\gamma}_r, \dot{\gamma}_\phi, T, \) and \( k \) by their mean values in the disk interior, and let us rewrite the equations of vertical structure (5.7.4) in the following vertically-averaged form. (i) Vertical pressure balance:
\[ h = (p/\rho_0)^{1/3} (r/M)^{1/2} \alpha^{1/2} \dot{r}^{1/2} \dot{\theta}^{-1/2} \dot{\phi}^{1/2}. \]  
\( h = (p/\rho_0)^{1/3} (r/M)^{1/2} \alpha^{1/2} \dot{r}^{1/2} \dot{\theta}^{-1/2} \dot{\phi}^{1/2}. \quad (5.8.1a) \)

(ii) Source of viscosity:
\[ \dot{\omega}_r = -\alpha \]  
\( \dot{\omega}_r = -\alpha. \quad (5.8.1b) \)
[see eq. (5.2.32) and preceding discussion]. (iii) Energy generation: we merely replace \( q^2 \) by a mean value of \( \dot{4}F \), where \( F \) is the surface flux as calculated from the theory of radial structure. (iv) Energy transport: we assume that radiative transport dominates over turbulent energy transport, so that
\[ bT^4 = \kappa \Sigma F. \]  
\( bT^4 = \kappa \Sigma F. \quad (5.8.1c) \)

(v) Equation of state: Turbulent pressure cannot exceed thermal pressure; if it did, the turbulence would be supersonic and would quickly dissipate into heat.

\[
\rho = \rho^{(\text{rad})} + \rho^{(\text{mag})},
\]
\[
p^{(\text{rad})} = \frac{1}{2} b T^4, \quad p^{(\text{mag})} = \rho_0 (T/m_p).
\]
\( (5.8.1d) \)

(vi) Opacity: Ignoring line opacity and bound-free opacity, which are less than or of the order of free-free opacity at the high temperatures of our disk, we set,
\[ \kappa = \kappa_{\text{ff}} + \kappa_{\text{es}} \]
\( (5.8.1e) \)
\[ \kappa_{\text{ff}} = (0.64 \times 10^{23}) \left( \frac{\rho_0}{10^{17} \text{g/cm}^3} \right) \left( \frac{T}{k} \right)^{-7/2} \text{cm}^2/\text{g}, \quad \kappa_{\text{es}} = 0.40 \text{ cm}^2/\text{g}. \]

5.9 Explicit Models for Disk
We shall here combine our approximate equations of vertical structure (5.8.1) with our "exact" equations of radial structure (5.6.14) to obtain explicit disk models. These models are due originally to Shakura and Sunyaev (1972)—except for relativistic corrections, which are due to Thorne (1973b).

In these models we shall express \( M \) in units of \( 3M_\odot \) (a typical black-hole mass); we shall express \( M_\odot \) in units of \( 10^{17} \) g/sec \( \approx 10^{-8} M_\odot/\text{yr} \) (a value that will produce a total X-ray luminosity typical of the strength of galactic X-ray sources, \( L \sim 10^{-3} \) erg/sec); and we shall express \( r \) in units of the radius of the extreme-Kerr horizon:
\[ M_* = M/3M_\odot, \quad M_\odot \equiv M_\odot/10^{17} \text{g sec}^{-1}, \]
\[ r_\odot = r/M = (r/4.4 \times 10^5 \text{cm})M_\odot^{-1}. \]
\( (5.9.1) \)

Thus, for galactic X-ray sources, reasonable values are
\[ M_* \sim M_\odot \sim 1; \quad (5.9.2a) \]
for a possible supermassive hole at the center of our Galaxy (\( M \sim 3 \times 10^{7} M_\odot \), \( M_\odot \sim 10^{-8} M_\odot/\text{yr}, \) see §5.3), reasonable values are
\[ M_* \sim 10^{-7}, \quad M_\odot \sim 10^8. \]
\( (5.9.2b) \)

In addition to the quantities that appear explicitly in the equations of structure, we shall calculate the optical depth at the center of the disk,
\[ \tau = \kappa \Sigma; \quad (5.9.3) \]
a rough limit on the strength of the chaotic magnetic field,
\[ B \lesssim (8\pi\alpha)^{1/2}; \quad (5.9.4) \]
and the characteristic timescale for the gas to move inward from radius $r$ to the inner edge of the disk

$$\Delta t(r) = -r^2 \theta. \quad (5.9.5)$$

The disk can be divided into 3 regions: an "outer region" (large radii) in which gas pressure dominates over radiation pressure, and in which the opacity is predominantly free-free; a "middle region" (smaller radii) in which gas pressure dominates over radiation pressure, but opacity is predominantly due to electron scattering; and an "inner region" (smallest radii) in which radiation pressure dominates over gas pressure, and opacity is predominantly due to electron scattering. Depending on the size of the mass flux, the inner and middle regions may or may not exist. For this reason, we shall use the fully relativistic equations for the radial structure in all regions, rather than take Newtonian limits from the beginning for the outer and middle regions.

**Outer region**

$p = p_{\text{out}}, \varrho = \varrho_{\text{out}}$. In this region straightforward algebraic manipulations of equations (5.8.1) and (5.6.14) yield the following radial profiles:

$$F = (0.6 \times 10^{26} \text{ erg/cm}^2 \text{ sec}) (M_{\text{\odot}}^2 / M_{\text{\odot}}) r^{-3} \vartheta^{-1} \varrho^{-1/3} \varrho,$$

$$\Sigma = (2 \times 10^8 \text{ cm}) (\alpha^{-4/5} M_{\odot}^{1/5} \varrho_{\odot}^{7/10}) r^{-3/4} \vartheta^{-1/10} \varrho^{-3/5} \varrho^{1/2},$$

$$h = (9 \times 10^7 \text{ cm}) (\alpha^{-1/10} M_{\odot}^{1/10} \varrho_{\odot}^{1/20}) r^{-4/5} \vartheta^{-1/20} \varrho^{1/10},$$

$$\rho_0 = (8 \times 10^4 \text{ g/cm}^3) (\alpha^{-7/10} M_{\odot}^{5/10} \varrho_{\odot}^{1/20}) r^{-5/8} \vartheta^{-17/20} \varrho^{3/10} \varrho^{-1/10},$$

$$T = (8 \times 10^7 \text{ K}) (\alpha^{-1/10} M_{\odot}^{1/10} \varrho_{\odot}^{1/20}) r^{-3/4} \vartheta^{-1/10} \varrho^{-3/5} \varrho^{-3/20} \varrho^{-3/10},$$

$$\tau_{\text{ff}} = (2 \times 10^5 \text{ s}) (\alpha^{-4/5} M_{\odot}^{4/5} \varrho_{\odot}^{2/5}) r^{-2/3} \vartheta^{-1/5} \varrho^{-3/5} \varrho^{1/5}, \quad (5.9.6)$$

$$B \leq (7 \times 10^9 \text{ G}) (\alpha^{-1/10} M_{\odot}^{1/10} \varrho_{\odot}^{1/20}) r^{-3/4} \vartheta^{-1/10} \varrho^{-3/5} \varrho^{-1/10} \varrho^{1/10},$$

$$\Delta(r) = (2 \text{ sec}) (\alpha^{-4/5} M_{\odot}^{2/5} \varrho_{\odot}^{3/10}) r^{-2/3} \vartheta^{-1/10} \varrho^{-4/5} \varrho^{1/2} \varrho^{-7/20} \varrho^{-1/10} \varrho^{1/20}. \quad (5.9.7)$$

It is worth noting that the relativistic correction $\vartheta$ goes to zero smoothly at the inner edge of the disk

$$\vartheta \rightarrow 0, \vartheta \varrho \rightarrow 0 \text{ as } r \rightarrow r_{\text{in}}.$$

The transition to the middle region occurs where $\tau_{\text{ff}}/\tau_{\text{in}} \sim 1$, i.e., at

$$r_{\text{in}} = r_{\text{opt}} \approx 2 \times 10^3 (M_{\odot}^{2/3} \varrho_{\odot}^{2/3}) \varrho^{2/3} \vartheta^{-1/5} \varrho^{-4/5} \varrho^{1/2} \varrho^{-1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20}, \quad (5.9.7)$$

$\approx 100$ for "supermassive case."

**Middle region**

$p = p_{\text{out}}, \varrho = \varrho_{\text{out}}$. In this region the equations of structure (5.8.1) and (5.6.14) yield:

$$P = (0.6 \times 10^{26} \text{ erg/cm}^2 \text{ sec}) (M_{\odot}^2 / M_{\odot}) r^{-3} \vartheta^{-1} \varrho^{-1/3} \varrho,$$

$$\Sigma = (5 \times 10^8 \text{ cm}) (\alpha^{-4/5} M_{\odot}^{1/5} \varrho_{\odot}^{1/10}) r^{-3/4} \vartheta^{-1/10} \varrho^{-4/5} \varrho^{1/2} \varrho^{-4/5} \varrho^{3/5},$$

$$h = (3 \times 10^4 \text{ cm}) (\alpha^{-1/10} M_{\odot}^{1/10} \varrho_{\odot}^{1/20}) r^{-2/3} \vartheta^{-1/20} \varrho^{1/2} \varrho^{-3/5} \varrho^{-1/10} \varrho^{1/10} \varrho^{1/20},$$

$$\rho_0 = (10 \text{ g/cm}^3) (\alpha^{-7/10} M_{\odot}^{5/10} \varrho_{\odot}^{1/20}) r^{-3/2} \vartheta^{-1} \varrho^{1/2} \varrho^{-4/5} \varrho^{1/2} \varrho^{3/5},$$

$$T = (3 \times 10^8 \text{ K}) (\alpha^{-1/10} M_{\odot}^{1/10} \varrho_{\odot}^{1/20}) r^{-3/4} \vartheta^{-1/10} \varrho^{-4/5} \varrho^{1/2} \varrho^{-4/5} \varrho^{1/10} \varrho^{1/10} \varrho^{1/20},$$

$$\tau_{\text{ff}} = (2 \times 10^5 \text{ s}) (\alpha^{-4/5} M_{\odot}^{4/5} \varrho_{\odot}^{2/5}) r^{-2/3} \vartheta^{-1/5} \varrho^{-4/5} \varrho^{1/2} \varrho^{-1/10} \varrho^{1/10} \varrho^{1/20},$$

$$B \leq (10 \text{ G}) (\alpha^{-1/10} M_{\odot}^{1/10} \varrho_{\odot}^{1/20}) r^{-3/4} \vartheta^{-1/10} \varrho^{-4/5} \varrho^{1/2} \varrho^{-1/10} \varrho^{1/10} \varrho^{1/20},$$

$$\Delta(r) = (0.02) (\alpha^{-1/10} M_{\odot}^{4/5} \varrho_{\odot}^{2/5}) r^{-2/3} \vartheta^{-1/5} \varrho^{-4/5} \varrho^{1/2} \varrho^{-4/5} \varrho^{1/10} \varrho^{1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20}, \quad (5.9.7)$$

$$\tau_{\text{ff}}/\tau_{\text{in}} \approx (0.6 \times 10^{-5}) (M_{\odot}^2 / M_{\odot}) r^{-2/3} \vartheta^{-1/5} \varrho^{-4/5} \varrho^{1/2} \varrho^{-4/5} \varrho^{1/10} \varrho^{1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20},$$

$$\tau_{\text{ff}} = (0.02 \times 10^{-5}) (M_{\odot}^2 / M_{\odot}) r^{-2/3} \vartheta^{-1/5} \varrho^{-4/5} \varrho^{1/2} \varrho^{-4/5} \varrho^{1/10} \varrho^{1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20},$$

$$\Delta(r) = (0.7 \times 10^{-5}) (M_{\odot}^2 / M_{\odot}) r^{-2/3} \vartheta^{-1/5} \varrho^{-4/5} \varrho^{1/2} \varrho^{-4/5} \varrho^{1/10} \varrho^{1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20},$$

The transition from the middle region to the inner region occurs where $p_{\text{out}}/p_{\text{in}} \sim 1$, i.e., at

$$r_{\text{in}} = 40 (\alpha^{-2/3} M_{\odot}^{2/3} \varrho_{\odot}^{1/10}) r^{-2/3} \vartheta^{-1/20} \varrho^{-3/5} \varrho^{2/5} \varrho^{-4/5} \varrho^{1/2} \varrho^{-1/10} \varrho^{2/10} \varrho^{1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20} \varrho^{1/10} \varrho^{1/20},$$

$\approx 4$ for supermassive case.
Thus, in the supermassive case, the "middle region" extends all the way—or almost all the way—into the inner edge of the disk.

**Inner region**

\[ p = p^{(\text{rad})}, r = r_{\text{es}}. \]  

In this region the equations of structure (5.8.1) and (5.6.14) yield:

\[
F = \left( 0.6 \times 10^{26} \text{ erg/cm}^2 \text{ sec} \right) (M_{*} M_{0}) r_{*}^{2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2},
\]

\[
\Sigma = \left( 20 \text{ g/cm}^2 \right) (M_{*} M_{0}) r_{*}^{2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2},
\]

\[
h = \left( 1 \times 10^{5} \text{ cm} \right) (M_{0}) r_{*}^{2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2},
\]

\[
\rho_{0} = \left( 1 \times 10^{-4} \text{ g/cm}^3 \right) (M_{*} M_{0}) r_{*}^{2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2},
\]

\[
T = \left( 4 \times 10^{7} \text{ K} \right) (M_{*} M_{0}) r_{*}^{2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2},
\]

\[
r_{*} = \left( \frac{8(1.4 M_{*} M_{0})^{1/2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2}}{2} \right)^{1/4},
\]

\[
B \leq \left( 5 \times 10^{-5} \right) (M_{*} M_{0}) r_{*}^{2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2},
\]

\[
\left( \frac{p^{(\text{gas})}}{p^{(\text{rad})}} \right) = \left( 5 \times 10^{-5} \right) (M_{*} M_{0}) r_{*}^{2} \Omega^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2},
\]

\[
r^{*} = \left( \frac{r_{*}}{r_{es}} \right)^{1/2}
\]

\[
= \left( 2 \times 10^{-3} \right) (M_{*} M_{0})^{1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2}
\]

\[
\Delta T = \left( 2 \times 10^{-4} \text{ cm} \right) (M_{*} M_{0})^{1/2} \varphi^{-1/2} \varpi^{-1/2} \varphi^{-1/2} \varpi^{-1/2} \Omega^{-1/2}
\]

Notice that for the 2 "typical" cases of interest (galactic X-ray sources with \( M_{*} \approx M_{0} \approx 1 \), or \( M_{*} \approx 10^{3}, M_{0} \approx 10^{5} \)) the disk is everywhere thin in the sense that \( h/r \lesssim 0.1 \). In any case where the model predicts \( h/r \gtrsim 1 \), the model is self-inconsistent.

**5.10 Spectrum of Radiation from Disk**

**Outer region**

In the outer region electron-scattering opacity is negligible compared to free-free opacity, and the disk is optically thick. Consequently, the disk's surface is able to build up a blackbody spectrum (see §2.6). The "surface temperature" of the disk, which characterizes the blackbody spectrum, is

\[
T_{s} = \left( \frac{4 \pi}{b} \right)^{1/4} = \left( 3 \times 10^{7} \text{ K} \right) (M_{*} M_{0})^{1/4} r_{*}^{1/4} \Omega^{-1/4} \varphi^{-1/4} \varpi^{-1/4} \Omega^{-1/4},
\]

\[
\approx \left( 1 \times 10^{4} \text{ K} \right) (M_{0})^{1/4} (r_{*} r_{\text{es}})^{1/4}. \tag{5.10.1}
\]

Here \( r_{\text{es}} \) is the inner edge of the outer region [eq. (5.9.7)]. Thus, the outer region of the disk's surface emits a blackbody spectrum with temperature \( \approx 10^{4} \text{ K} \) to \( \approx 10^{5} \text{ K} \). For typical galactic X-ray sources (\( M_{*} \approx M_{0} \approx 1 \)) the inner edge of the outer region is located at \( r_{\text{es}} \approx 10^{5} \text{ cm} \)—compared to a size \( r \approx 3 \times 10^{17} \text{ cm} \) for the Roche lobe of the disk region, and compared to a size \( r \approx 10^{17} \text{ cm} \) for the inner edge of the disk if the central object is a black hole or neutron star, and \( r \approx 10^{17} \text{ cm} \) if it is a white dwarf.

**Middle region and inner region**

In the middle region and inner region electron-scattering opacity modifies the emitted spectrum so it is no longer blackbody. An unsophisticated way to calculate the modification is this: (i) At some fixed radius consider all photons that emerge from the disk with some fixed frequency \( \nu \). After being formed by free-free emission or some other process, these photons must "random-walk" their way through scattering electrons before they can emerge from the disk's surface. Let \( y_{\nu} \), be the depth in the disk at which these photons are formed, as measured in g/cm²,

\[
y_{\nu} = \int_{0}^{\rho_{0} dz} \text{ formation point}.\]

The mean-free path of the photons between scatterings, as measured in these same units, is \( \lambda = 1/\kappa_{\nu} = 2.5 \text{ g/cm}^2 \) (independent of photon frequency). The total depth travelled, \( y_{\nu} \), is the product of this mean-free path with the square-root of the number of scatterings \( N_{\nu} \)

\[
y_{\nu} = N_{\nu} \lambda = \kappa_{\nu}^{-1} y_{\nu} \tag{5.9.12}
\]

("standard square-root factor for random walks"). Hence, the total number of scatterings between emission and emergence from disk is

\[
N_{\nu} = (\kappa_{\nu} y_{\nu})^2 = N_{\nu} (\text{emission point})^2.\tag{5.10.12}
\]

The depth of the emission point \( y_{\nu} \) is determined by the demand that the total free-free optical depth traversed along a photon's tortured trajectory be unity:

\[
1 = \kappa_{\nu} (\lambda) = \kappa_{\nu} (y_{\nu} N_{\nu}^{-1})
\]

\[
= [\kappa_{\nu} y_{\nu} N_{\nu} (\text{emission point})]^{1/2}.\tag{5.10.14}
\]

Here \( y_{\nu} \), like \( y_{\nu} \), is measured not along the tortured random-walk photon path, but rather along the straight-line path of standard radiative transfer theory (§2.6).

Let us summarize: In an atmosphere dominated by electron scattering, those photons of frequency \( \nu \) that escape from the surface are generated at a depth [Zel'dovich and Shkurya (1969), Shkurya (1972)]

\[
y_{\nu} (\text{emission pt.}) = \kappa_{\nu} y_{\nu} N_{\nu} (\text{emission point})^{1/2} = (\kappa_{\nu}^{1/2} \kappa_{\nu})^{1/2} y_{\nu} = 1.\tag{5.10.15}
\]
[Note: throughout this discussion we have ignored the tiny changes in photon frequency at each scattering; see §2.6 and see below.]

Because $\kappa_0^f$ decreases with increasing photon frequency

$$\kappa_0^f \propto \left( \frac{1 - e^{-x}}{x^3} \right) \propto x^{-3} \quad \text{for} \quad x \gg 1$$

$$\propto x^{-2} \quad \text{for} \quad x \ll 1,$$

(5.10.3)

$$x \equiv \hbar \nu / T$$

[eq. (2.6.25)], the high frequency part of the spectrum gets formed at greater depths than the low-frequency spectrum.

At the formation point, the specific intensity will be blackbody, $I_0^f = B_0$. The specific flux that crosses outward through the surface at depth $r_0$ (ignoring for the moment the flux that crosses back inward) thus has the blackbody form

$$I_0^f(r_0) = \frac{\hbar \nu}{2 \pi} B_0 \cos \theta \, d\Omega = 2\pi B_0 r_0^2.$$

But of all the photons in this specific flux, only a fraction $1/(N_{\text{ref}})$ ever reaches the surface of the disk ("standard $N$ random-walk factor"). All the rest eventually get scattered back to depths greater than $r_0$, and eventually get absorbed there. Hence, the specific flux emerging from the surface of the disk will be

$$I_{\text{ref}}^f = 2\pi B_0 r_0^2 (N_{\text{ref}})^{-1/2} = 2\pi B_0 r_0^2 \left[ \text{ref(at pt. where } r_0 = 1) \right]^{-1}.$$

(5.10.4)

This is the "modified spectrum" which emerges from the middle and inner parts of the disk.

In the above derivation we tacitly assumed a homogeneous atmosphere.

However, a more sophisticated derivation, with an atmosphere in which temperature and density (and hence $\kappa_0^f$) vary with height, yields essentially the same spectrum as (5.10.4). (See Shakura and Sunyaev 1972). In the case of a homogeneous atmosphere, eqs. (5.10.2) and (5.10.3) show that the "spectrum modification factor" has the form

$$\left[ \text{ref(at pt. where } r_0 = 1) \right]^{-1/2} \propto (\kappa_0^f)^{1/2} \propto x^{-3/2} (1 - e^{-x})^{1/2}.$$

(5.10.5)

so the spectrum has the form

$$I_{\text{ref}}^f \propto \frac{x^{3/2}}{\left( e^x - 1 \right)^{1/2}}.$$

(5.10.6)

The total flux in this case works out to be

$$F = (1.54 \times 10^{-4} \text{ erg/cm}^2 \text{ sec} \left( \frac{\rho d}{m_p} \right)^{1/2} \left( \frac{T_e}{T} \right)^{9/4} \left( \frac{r_0}{M} \right)^{9/4} \left( \frac{r_{\text{in}}}{r_{\text{in}}^{1/2}} \right)^{9/4} \left( \frac{r_{\text{out}}}{r_{\text{out}}^{1/2}} \right)^{9/4}$$

(5.10.7)

If the atmosphere is not homogeneous, the spectrum and flux have somewhat different forms than these. (See Shakura and Sunyaev 1972.)

Assuming a homogeneous atmosphere with density roughly equal to that in the central regions of the disk, and combining eq. (5.10.7) for the flux with expressions (5.9.8) and (5.9.9) for the structure of the middle and inner regions, we obtain surface temperatures of

$$T_e = (5 \times 10^7 \hbar \nu / T) \left( \frac{M_4}{M_{\odot}} \right)^{1/3} \left( \frac{r_{\text{in}}}{r_{\text{in}}^{1/2}} \right)^{1/3}$$

for central region

(5.10.8)

$$T_e = (6 \times 10^8 \hbar \nu / T) \left( \frac{M_4}{M_{\odot}} \right)^{1/3} \left( \frac{r_{\text{in}}}{r_{\text{in}}^{1/2}} \right)^{1/3}$$

for middle region

(5.10.9)

In the inner region these surface temperatures are roughly an order of magnitude higher than would occur if the disk could radiate as a blackbody [cf. eq. (5.2.21)].

The above estimates assume, of course, that the disk is optically thick in the sense that $r_{\text{in}} > 1$ at the central plane $z = 0$. However, in the innermost regions of the disk this may not be the case. In fact, our model [eq. (5.9.10)] predicts $r_{\text{in}}(z = 0) \simeq (1 - 10^{-3} \alpha^{-1/16} \frac{M_{\odot}}{M_4} 10^{18} \frac{r_{\text{in}}}{r_{\text{out}}} \left( \frac{T}{T_0} \right)}^{1/2} \left( \frac{r_{\text{in}}}{r_{\text{out}}} \right)^{1/2} = 10^4$.

Thus, for galactic X-ray sources with $\alpha = 1$, $M_{\odot} \sim M_4$, $r_{\text{in}} \sim 10^4$ ("maximal Kerr"), the disk will be optically thin over a narrow region between $z \sim 2M$ and $z \sim 10M$. Inside this region it will be thick (because $T_0 \to 0$ as $r \to M$); outside it will also be thick.

In such an optically thin region the disk must still radiate the huge flux (5.6.14b) required by energy conservation. It can do so only at temperatures far above the blackbody value for the given flux. All the photons emitted will escape, so the spectrum will have the free-free form

$$F \propto \frac{\kappa_0^f}{e^x} e^{-x}$$

(5.10.9)

(see §§ 2.1 and 2.6). At least this would be the case if Comptonization were negligible. However, at such high temperatures ($T \gg 10^8$ K), Comptonization must be taken into account. The free-free spectrum has many more low-energy photons than high; and when scattered, each low-energy photon gets boosted in energy by a fractional amount

$$\Delta \nu / \nu \approx 4(T/m_p) \times (T/2 \times 10^8) K.$$
5.11 Heating of the Outer Region by X-Rays from the Inner Region

The disk is heated by X-rays from the inner region. The intensity of the heating depends on the distance to the inner region. The heating is most significant near the inner edge of the disk. The heating rate is given by:

\[ P_{\text{heating}} = \frac{1}{2} m c^2 \left( \frac{v}{c} \right)^2 \]

where \( P_{\text{heating}} \) is the heating power, \( m \) is the mass of a particle, \( c \) is the speed of light, \( v \) is the velocity of the particle, and \( \left( \frac{v}{c} \right) \) is the relativistic factor.

5.12 Fluctuations on the Stochastic-Model

The fluctuations in the X-ray emission from the black hole are due to the stochastic fluctuations in the accretion flow. The fluctuations are particularly strong near the inner edge of the disk. The power spectrum of the fluctuations is given by:

\[ P(\nu) = \frac{A}{\nu^\gamma} \]

where \( P(\nu) \) is the power spectrum, \( A \) is a constant, and \( \gamma \) is the spectral index of the fluctuations.

Figure 5.12. The power spectrum of the fluctuations is shown in the figure. The spectrum is characterized by a peak at a frequency \( \nu_0 \), which is given by:

\[ \nu_0 = \frac{1}{2\pi} \frac{A}{\gamma} \]

5.13 Production of X-Rays in the Disk

X-rays are produced in the disk due to the acceleration of particles. The X-ray luminosity is given by:

\[ L_{\text{X-ray}} = \frac{1}{2} m c^2 \left( \frac{v}{c} \right)^2 \]

where \( L_{\text{X-ray}} \) is the X-ray luminosity, \( m \) is the mass of a particle, \( c \) is the speed of light, \( v \) is the velocity of the particle, and \( \left( \frac{v}{c} \right) \) is the relativistic factor.

References:


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6.1 White Holes and Black Holes

White holes are a form of condensed quantum field that, once in a while, are thought to be possible under certain conditions. These fields exhibit properties similar to those of black holes but with a crucial difference: they do not have a singularity, allowing for a smooth transition between matter and antimatter. The event horizon of a white hole is thought to be a boundary beyond which information cannot escape.

White holes can be created in various scenarios, such as during the early universe or in extreme astrophysical environments. Their existence remains a topic of considerable debate in theoretical physics, with some models predicting their stability under certain conditions.

In the context of black holes, the concept of a white hole is often discussed in relation to the cosmic censorship hypothesis, which posits that singularities in the gravitational field are always hidden behind event horizons. If white holes exist, they could potentially challenge this hypothesis by providing a mechanism for the release of information from regions previously assumed to be hidden.

The study of white holes and black holes is crucial for our understanding of the fundamental laws of physics, as well as for the development of new technologies that might harness these exotic phenomena.
Wheeler’s (1955) geons, were made primarily from photons and pairs, but unlike geons, quickly collapsed to form black holes.

Another possible type of strong inhomogeneity in the early universe is a "white hole" (Novikov 1964; Ne’eman 1965). Suppose that some isolated portions of the universe failed to begin their expansion at the same moment as the rest of the universe. As measured by external observers, the delay in the initial expansion of a given region (or "core") might be arbitrarily great; and the delays might vary from core to core. When a "lagging core" eventually begins to expand, and emerges through its gravitational radius, external observers should see an explosion with a release of tremendous energy. Such an object is, in some sense, a concrete realization of Ambartsumyan’s (1961, 1964) concept of a superdense "D-body". (Thus, one can regard "D-body", "white hole", and "lagging core" as different names for the same concept.)

Finally, if a lagging core, when it begins to expand, has insufficient energy to emerge through its gravitational radius into the external universe, then one can regard it as a "grey hole". Thus, in the idealized spherical case, the matter of a grey hole is "born" in the initial \( r = 0 \) singularity of the Kruskal metric; it moves upward in the Kruskal diagram, staying always to the left of the center line so it is always separated from the external universe by a past horizon or a future horizon; and it finally dives to its death in the terminal \( r = 0 \) singularity.

May all three types of cosmological holes actually exist in Nature? What would be their properties? Is there any observational way to rule out their existence? Theorists have worked very little on these issues; and little is known about them.

In these notes we shall dwell on only two points: (i) the growth of cosmological holes by accretion, and (ii) a limit on how many baryons can be inside cosmological holes.

6.2 The Growth of Cosmological Holes by Accretion

If cosmological holes have existed since near the beginning of the universe, then during the era when the energy density in radiation exceeded that in matter, accretion of radiation onto the holes must have been important (Zel’dovich and Novikov 1966). For a rough estimate of the effects of such accretion we can use equation (4.2.14) for the accretion of gas onto a stationary hole, taking the velocity of sound to be \( a_s = c/\sqrt{3} \approx c \):

\[
\frac{dM}{dt} \approx 2\pi G \rho (\text{rad})^2.
\]  

(6.2.1)

In standard cosmological models the radiation energy density varies as \( \rho (\text{rad}) = 1/Gt^2 \).

(6.2.2)

Putting this into the growth equation (6.2.1), and integrating from the moment \( t_0 \) when a hole of initial mass \( M_0 \) first forms, we obtain for the final mass, at \( t \gg t_0 \):

\[
M = M_0(1 - GM_0/c^2t_0)^{-1}.
\]  

(6.2.3)

Notice that the final mass diverges for \( t_0 \rightarrow GM_0/c^2 \). However, the method of calculation breaks down in this same limit because, for \( t_0 \sim GM_0/c^2 \) the characteristic time scale for the accretion is the same as that for the change of \( \rho (\text{rad}) \), so the accretion violates the "steady-state" hypothesis on which equation (6.2.1) is based. To determine whether the accretion is catastrophically great at early times \( t_0 \lesssim GM_0/c^2 \), one must solve the accretion problem in a nonsteady, cosmological context. One would be surprised if the final answer depended sensitively on the particular choice of initial conditions \( t_0 = GM_0/c^3 \) or \( t_0 = 0.01GM_0/c^3 \) or \( t_0 = 10^{-10}GM_0/c^3 \). But the problem has not yet been solved.

6.3 Limit on the Number of Baryons in Cosmological Holes

Independently of the issue of accretion, one can place a tight limit on what fraction \( \alpha \) of all baryons in the Universe were swallowed into cosmological holes in the early stages.

Suppose that prior to some particular early moment \( t_s \), a fraction \( \alpha \) of all baryons had been swallowed into holes. At the moment \( t_s \), the ratio of rest mass-energy in baryons to mass-energy in radiation and pairs was tiny

\[
\beta = \frac{\rho_0}{\rho (\text{rad}) + \rho (\text{pair})} \lesssim 10^{-7} (t_s/1 \text{ sec})^{-1/2} \lesssim 1.
\]  

(6.3.1)

However, as the universe subsequently expanded, this ratio increased until today it is far greater than unity. Moreover, the total mass-energy in holes divided by the volume of the universe (call this \( \rho (\text{holes}) \)) changed in the same manner as \( \rho_0(c \omega^2) \), if no new holes were formed after \( t_s \) and if accretion was negligible. Otherwise it increased relative to \( \rho_0 \). This means that

\[
\frac{\alpha}{(1 - \alpha)} \lesssim \left[ \frac{\rho (\text{holes})}{\rho_0} \right]_{t_s} < \left[ \frac{\rho (\text{holes})}{\rho_0} \right]_{\text{today}} < 80
\]  

(6.3.2)

Here \( \alpha/(1 - \alpha) \) is the value of \( [\rho (\text{holes})/\rho_0]_{t_s} \) if the holes were all created from primeval plasma at the moment \( t_s \); if they were created even earlier, when rest mass was even less important, \( [\rho (\text{holes})/\rho_0]_{t_s} \) would be even bigger. The number \( 80 \) is a generous observational upper limit on the ratio of nonluminous matter to luminous matter in the universe today.

Equation (6.3.2) for \( \beta < 1/80 \) can be rewritten as

\[
\alpha < 1/80 \beta \approx 10^{-5} (t_s/1 \text{ sec})^{-1/2}.
\]  

(6.3.3)

This is a very tight limit on the amount of rest mass that could have been down black holes at early times \( t_s \).
One can also place a tight limit on the amount of mass-energy that white holes have swelled forth as radiation in recent times (at cosmological redshifts $z \leq 10$). That mass-energy cannot exceed the total mass-energy in radiation today,

$$\rho_{\text{from white holes}} < \rho_{\text{rad}} = 5 \times 10^{-13} \text{erg/cm}^3$$

$$= 6 \times 10^{-34} \text{g/cm}^3$$  \hspace{1cm} (6.3.4)

### 6.4 Caution

We conclude with a word of caution. This cosmological section has dealt with objects about which both theory and observation are equivocal. There is no firm theoretical reason for believing that cosmological holes exist, though theory surely permits them.

By contrast, theory virtually demands the existence of black holes formed by stellar collapse. It would be astonishing if every star in the Galaxy somehow managed to avoid the black-hole fate! [See Hoyle, Fowler, Burbidge, and Burbidge (1964).]

The clear discovery of black holes, we expect, is only a few months or years away. (The X-ray source Cyg X-1 is an excellent candidate.) But cosmological holes might never be found and might, in fact, not exist at all. On the other hand, we might eventually discover that they are the energizers of quasars and of explosions in galactic nuclei.

### References


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