Problem Set 8

Mar 5, 2004 — Due Mar 10, 2004

ACM 95b/100b — 3pm at Firestone 303

E. Sterl Phinney — (2 pts) Include grading section number

Useful Readings: For Green's functions, see class notes and refs on PS7 (esp Carrier and Pearson section 7.4 and 3/5/04 class notes). Bessel functions and Legendre functions: Arfken Chapters 11 and 12, Carrier and Pearson Chapter 11. Hassani is not quite so useful as Arfken on this topic, but Chapter 12 and sections 15.3, 7.3, 7.4 and 7.6 come closest.

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1. (10 points) Consider the same DE as in PS7, #4, but modify the upper boundary condition to be inhomogeneous: $y'(1) = 1$ (so that the problem is no longer of Sturm-Liouville form), i.e. consider

$$\frac{d^2 y}{dx^2} + \lambda^2 y = f(x), \quad y(0) = 0, \quad y'(1) = 1 \tag{1}$$

Still assuming λ is not an eigenvalue of the homogenous problem (PS7, #4), give the solution to eq (1) in terms of the Green's function you found in PS7 #4b. [hint: You can do this in two ways. One was given in class 3/5/04. Another is to let $y = h(x) + q(x)$ where $q(x)$ is chosen so that h satisfies an ODE like eq (1) with homogeneous boundary conditions, but a modified $f(x)$.]

In addition to Taylor series representations of the solutions of ordinary differential equations which you have already encountered, there are three very useful alternative ways of representing solutions: integral representations, generating functions and recursion relations.

The problems below introduce you to the power of these types of representation for the solutions of Bessel's equation (which arose in PS6 # 3 when you separated ∇^2 in cylindrical polar coordinates) and Legendre's equation (which arose in PS6 #2c when you separated ∇^2 in spherical polar coordinates, with $x = \cos\theta$).

In solving all these problems you may assume (as can be proven) that all the integrals and infinite sums that appear are sufficiently convergent that it is allowed to interchange differentiation, integration and summation at will.

2. (5×7 points)

a) Show that

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos[x\sin\theta - n\theta]\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \exp[i(x\sin\theta - n\theta)]\, d\theta \tag{2}$$

is a solution[1] of Bessel's equation of order n:

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \tag{3}$$

where primes indicate differentiation with respect to x. [hint: one step along the way should be to show that if $u(x, \theta) \equiv (1/2\pi)\exp[i(x\sin\theta - n\theta)]$, and $y(x) = \int_0^{2\pi} u\, d\theta$, then

$$x^2 y'' + xy' + (x^2 - n^2)y = \int_0^{2\pi} \left[-\frac{d^2 u}{d\theta^2} - i2n\frac{du}{d\theta} \right] d\theta \quad] \tag{4}$$

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[1]This is the integral representation of the Bessel functions. You already encountered the $n = 0$ case in PS7 #5.

b) Show that for integer $n \geq 0$ the functions defined by eq (2) have the following initial conditions at the singular point $x = 0$ of Bessel's equation:

$$J_0(0) = 1; \ J_n(0) = 0 \text{ for } n \neq 0; \quad J_1'(0) = 1/2; \ J_n'(0) = 0 \text{ for } n \neq 1 \tag{5}$$

c) Show that[2]

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{6}$$

[hint: work backwards: assume this is true, let $t = \exp(i\theta)$, and use orthogonality of $\exp[im\theta]$ on $[0, 2\pi]$ to evaluate the terms of the resulting Fourier series and recover eq (2).]

d) Show that[3]

$$e^{ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta}, \quad e^{ix \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(x)e^{in\theta} . \tag{7}$$

[hint: see part (c)]

e) Show that[4]

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x) \tag{8}$$

[hint: take $\partial/\partial x$ of the generating function equation eq (6), replace the remaining exponential by the right hand side, and equate the coefficients of like powers [why?] on both sides.]

3. (6×6 points)

a) Show that if there is no ϕ dependence, the equation for $\Theta(\theta)$ that you found by separation of variables in PS6,#2c reduces to

$$\sin \theta \frac{d}{d\theta}\left(\sin \theta \frac{d\Theta(\theta)}{d\theta}\right) + \lambda \sin^2 \theta \ \Theta(\theta) = 0 , \tag{9}$$

where λ is the separation constant.

b) Show that if you let $x = \cos \theta$, and $y(x) = \Theta(\theta)$, this equation becomes

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + \lambda y = 0 \tag{10}$$

This equation is of Sturm-Liouville form, and has regular singularities at $x = 1$, $x = -1$. It can be shown that there exist solutions that are finite at both $x = 1$ and $x = -1$ if and only if $\lambda = n(n + 1)$ for positive integer n. Equation (10) is then called *Legendre's equation of order n*[5].

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[2] This is the generating function for the Bessel functions.

[3] These equations are central to quantum scattering theory, diffractive optics (the expansion of plane waves), the theory of planetary perturbations and tides (periodic perturbations in a rotating frame), the theory of gravitational wave detection, and the theory of FM radio. Bessel functions can arise in situations having nothing to do with the separation of ∇^2 in cylindrical coordinates!

[4] This is one of the many recursion relations for Bessel functions, useful in numerically computing them, and evaluating their integrals.

[5] This and the related 'associated Legendre equation' [separation of PS6,#2c, keeping the φ dependence, expanded in $\exp(im\varphi)$] arises in quantum mechanics and chemistry [remember those s,p,d,f orbitals? They are $n = 0, 1, 2, 3$], spectroscopy, potential theory of electromagnetism and planets, theory of radiation of sound, elastic waves [e.g. earthquakes], electromagnetic waves and gravitational waves.

c) The solutions of Legendre's equation of order n which are finite at $x = \pm 1$ are called the Legendre polynomials (which we already introduced in lecture as the orthogonal polynomials of weight 1 on [-1,1] —cf PS5 #1,#3). It can be shown that they have the generating function

$$g(r, x) = \frac{1}{\sqrt{1 - 2rx + r^2}} = \sum_{n=0}^{\infty} P_n(x) r^n \tag{11}$$

for $|r| < 1$. Use this to show that

   i. $P_n(1) = 1$

   ii. $P_n(-1) = (-1)^n$

   iii. $P_0(x) = 1, \quad P_1(x) = x$

d) Equate terms in the expansion of $(1 - 2xr + r^2)\partial/\partial r$ of eq (11) to derive the recurrence relation $(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$ for $n \geq 1$. Use this to find $P_2(x)$.

e) It can be shown that another way to represent the Legendre functions is through Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n (x^2 - 1)^n \tag{12}$$

Show that for integer n, this is a polynomial of order n (and hence is finite at $x = \pm 1$, as desired), and show that the Legendre polynomials are orthogonal:

$$\int_{-1}^{1} P_n(x) P_m(x) \, dx = \frac{2}{2n + 1} \delta_{m,n} \tag{13}$$

[hint: integrate by parts. At some point you should encounter

$$(-1)^n \int_{-1}^{1} (x^2 - 1)^n \, dx = 2 \int_0^1 (1 - x^2)^n \, dx = 2^{2n+1} \text{Beta}(n+1, n+1) = 2^{2n+1} (n!)^2 / (2n+1)!] \tag{14}$$

f) In predicting the rate at which atoms will absorb or emit radiation in jumping between two quantum states with respective angular quantum numbers n and m, the following integral arises:

$$I = \int_{-1}^{1} x P_n(x) P_m(x) \, dx \tag{15}$$

Show that[6]

$$I = \frac{2(n + 1)}{(2n + 1)(2n + 3)} \delta_{m,n+1} + \frac{2n}{(2n + 1)(2n - 1)} \delta_{m,n-1} \tag{16}$$

[hint: use the results of the previous two parts]

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4. $(7 \times 3$ points extra credit) **Bessel Functions and FM radios** (Maple, Mathematica or other computer numerics/graphics needed)

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[6]This is an example of what in quantum mechanics is called a 'selection rule': the deltas in the integral show that there can be no strong (electric dipole) coupling between two energy levels unless the difference between their angular momenta is exactly one $\hbar$ unit.

FM (*Frequency Modulated*) radio works by encoding an audio signal $m(t)$ (air pressure as a function of time) as frequency modulation of a radio wave about a carrier frequency f_c [f_c is the frequency to which you 'set your FM dial']:

$$S_{FM}(t) \;=\; A_c \cos \Phi(t), \qquad (17)$$

$$\Phi(t) \;=\; 2\pi f_c t + 2\pi k \int_{-\infty}^{t} m(t) dt \qquad (18)$$

Note that the instantanous frequency of the signal is thus

$$f(t) = \frac{1}{2\pi}\frac{d\Phi}{dt} = f_c + km(t) \;, \qquad (19)$$

as implied by the name FM. We would like to understand two things: why are FM radio channels separated by 200kHz on the radio dial, and how should the constant k be chosen?

Consider for simplicity the case when the audio signal is a pure tone of frequency f_M: $m(t) = A_M \cos(2\pi f_M t)$, so

$$S_{FM}(t) = A_c \cos(2\pi f_c t + k A_M \sin(2\pi f_M t)) \qquad (20)$$

The product $k A_M \equiv \beta_f$ is known as the "*frequency modulation index*". From eq (19) we see that in this case

$$f(t) = f_c + f_M \beta_f \cos(2\pi f_M t) \;, \qquad (21)$$

so β_f is the ratio of the peak frequency deviation to the modulation frequency.

a) Show that $J_{-n}(x) = J_n(-x) = (-1)^n J_n(x)$.

b) Show that the Fourier transform of $S_{FM}(t)$ given by eq (20) is an equally spaced series of delta functions (called sidebands) with separation f_M, centered on f_c, and with amplitudes given by the Bessel functions [hint: use the result of Problem 2d]:

$$\tilde{S}_{FM} = A_c/2 \sum_{n=-\infty}^{\infty} J_n(\beta_f)\,(\delta(f - [f_c + nf_M]) + \delta(f + [f_c + nf_M])) \qquad (22)$$

c) Plot the amplitudes of the first 10 sidebands (i.e. for $n = 0, 1, \ldots, 10$) for $\beta_f = 0.3$, $\beta_F = 3$ and $\beta_f = 7$. Notice that the number of sidebands with significant amplitude is approximately $\beta_f + 1$, as one might have intuited (without the $+1$) from equation (21).

d) Show that

$$1 = \sum_{n=-\infty}^{\infty} J_n(x)^2 = J_0^2(x) + 2\sum_{n=1}^{\infty} J_n^2(x) \;. \qquad (23)$$

e) Show that the total power radiated in some very long time $2T$ is $\int_{-T}^{T} S_{FM}^2 dt = 2TA_c^2/2$ is equal to $4T$ times the sums of squares of the amplitudes of the sidebands. [hint: you can either use the result of Problem 2d, or Parseval's theorem, followed by the result of the previous part.]

f) CD quality music is sampled at 44.1kHz, so encodes frequencies up to 22kHz. Bad-sounding telephone and AM radio encodes only up to 3kHz. FM radio is intermediate in quality, so suppose it encodes frequencies up to 10kHz. We want to choose the constant k in equation (18), and hence β_f as large as possible so as to maximise the dynamic range of music we can encode even when the radio signal is weak (i.e. we can measure the amplitudes of the various sidebands only with relatively poor accuracy). Show that

for $f_M = 10$kHz, you have to choose $\beta_f < 8.6$ to ensure that less than 0.005 of the signal power leaks into the adjoining FM radio channels (whose sidebands start at ± 100kHz from the center channel: remember the stations are separated by 200kHz). [hint: you will need to evaluate the $1 - \sum_{-10}^{10} J_n^2(\beta_f)$ as a function of β_f].

g) Show that for a 3.33kHz signal encoded with the same $\beta_f = 8.6$, the fraction of the signal power leaking into the next FM radio channel is only 2×10^{-29}. Thus the owners of FM radio channels can trade between high dynamic range at low (talk show) frequencies, or lower dynamic range at high frequencies (instrument harmonics on music stations).

Total points: 83