Problem Set 7

February 28, 2003 — Due March 7, 2003
ACM 95b/100b — 3pm at Firestone 303
(2 pts) Include grading section number

Read: For series solutions, see BdP Chapter 5 and also look at Prof. Pierce's online notes. For separation of variables, read BdP Section 10.5; or Haberman Chapter 1, Sections 2.1–2.3; or Farlow Lessons 1–5.

1. (5 points each) For each of the following partial differential equations, what system of ordinary differential equations results when separation of variables is applied? (Make sure to clearly define your separation constants.)

   a) $u_{tt} = c^2 u_{xx}$ one-dimensional wave equation, c=wave speed
   b) $u_{xx} + u_{yy} + u_{zz} = 0$ Laplace's equation in Cartesian 3-space
   c) $u_t = k u_{xx} - c u_x$ one-dimensional advection-diffusion equation
   (k=diffusion coefficient, c=advection speed)
   d) $v_t + r x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = rv$ Black-Scholes PDE
   $v(x,t)$=stock option price, x=price of the option's underlying stock, t=time
   Constants: r=interest rate, σ= volatility of the underlying stock

2. (5 points each) What systems of ordinary differential equations arise when separation of variables is applied to the heat equation $u_t = \kappa \nabla^2 u$ in these geometries? (Make sure to clearly define your separation constants.)

   a) Cartesian 3-space (x, y, z)
   b) Polar coordinates with radial symmetry (no θ dependence)
   c) Polar coordinates
   d) Spherical coordinates with spherical symmetry (no θ and ϕ dependence)
   e) Spherical coordinates

   Note: In polar coordinates (r, θ),

   $$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

   and in spherical coordinates (r, θ, ϕ), where θ is the co-latitude and ϕ the longitude,

   $$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}.$$

3. (5 points each) Your ODE governing the r-dependence in problem 2c should be

   $$r^2 R''(r) + r R'(r) + (\lambda r^2 - n^2) R(r) = 0. \tag{1}$$

   (Your constants might have been chosen differently.)

   a) In the case of $\lambda = 0$, equation (1) is known as the equidimensional or Cauchy or Euler equation. Find the general solution to this equation for any integer n. (**Hint:** For $n \neq 0$, try substituting $R(r) = r^\alpha$ and solve for α.)
   b) Show that the change of variables $f(z) = R(r)$, $z = r\sqrt{\lambda}$, transforms (1) to Bessel's equation,

   $$z^2 f''(z) + z f'(z) + (z^2 - n^2) f(z) = 0. \tag{2}$$

   c) Since Bessel's equation has a regular singular point at $z = 0$, we use the method of Frobenius to obtain series solutions. Let

   $$f(z) = z^p \sum_{k=0}^{\infty} a_k z^k,$$

   and find the roots of the indicial equation. For $n > 0$, these roots should indicate that one linearly independent solution blows up at $z = 0$, another remains bounded.

d) Show that if n is an integer, the solution that is bounded at $z = 0$ is a multiple of

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!},$$

which is known as the nth order Bessel function of the first kind. (The other solution that blows up at $z = 0$ is known as the Bessel function of the second kind, $Y_n(z)$.)

4. (5 points each) Your ODE governing the θ-dependence in problem 2e should be

$$\sin\theta \frac{d}{d\theta}\left[\sin\theta \frac{d\Theta}{d\theta}\right] = (n^2 - \lambda \sin^2\theta)\Theta(\theta).$$

(Your constants might have been chosen differently.)

a) In the case of $n = 0$, show that the change of variables $y(x) = \Theta(\theta)$, $x = \cos\theta$ transforms the ODE above to Legendre's differential equation,

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0.$$

b) Does Legendre's equation have any singular points? (Are they regular or irregular?)

c) Since Legendre's equation has an ordinary point at $x = 0$, we can find a power series solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Show that

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+1)(k+2)} a_k.$$

d) Show that if the series does not terminate for some finite k, then the series must diverge at either $x = 1$ or $x = -1$. This means that if we seek solutions that are bounded at $x = \pm 1$, we must have $\lambda = m(m+1)$, for $m = 0, 1, 2, 3, \ldots$.

5. (10 points) Verify these orthogonality conditions for sines and cosines (assuming that m and n are nonnegative integers).

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \text{ or } m = n = 0 \\ L/2 & \text{if } m = n, \end{cases}$$

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{if } m = n \neq 0 \\ L & \text{if } m = n = 0 \end{cases}$$

(The addition formulas for sine and cosine will be helpful.) In addition, write down the orthogonality conditions for the sines and cosines over the range $[-L, L]$. (You don't have to rederive them, just use the symmetric properties of the integrands above.)

6. (25 points) Find the solution of the following boundary value problem.

$$\begin{aligned} \nabla^2 u = u_{xx} + u_{yy} &= 0 \qquad \text{in the rectangle } 0 \le x \le L, 0 \le y \le H \\ u_x(0, y) = u_x(L, y) &= 0 \\ u(x, 0) &= 0 \\ u(x, H) &= f(x) \end{aligned}$$

Your final answer should indicate how all constants can be obtained from $f(x)$.

Total points: 122