

## Problem Set 7

Feb 27, 2004  
ACM 95b/100b  
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Due Mar 5, 2004  
3pm at Firestone 303  
(2 pts) Include grading section number

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**Useful Readings:** For Fourier series and transformations: Arfken Chapters 14 and 15. Hassani Chapter 8. For Green's functions, Carrier and Pearson sections 7.2-7.4, Arfken section 9.5 (sections 16.5, 16.6 in earlier editions!), Hassani chapter 20 and sections 21.1-3. Notice that these use different definitions of the Fourier transform than the one given in class, where the FT of  $g(x)$  was defined as  $FT[g] \equiv \tilde{g}(f) = \int_{-\infty}^{\infty} g(x) \exp(-i2\pi fx) dx$ , with inverse  $g(x) = \int_{-\infty}^{\infty} \tilde{g}(f) \exp(i2\pi fx) df$ . This definition has the advantage that there are no factors of  $2\pi$  or  $\sqrt{2\pi}$  to keep track of, and you are encouraged to use it. If you use some other definition, be sure to state clearly what you are using.

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1. a) (8 points) Find the Fourier series representation of the period  $2\pi$  function  $f(x)$  for which

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 \leq x < \pi \end{cases} \quad (1)$$

- b) (6 points) What function does the sum of only the sine terms of your series in part (a) represent? What function does the sum of only the cosine terms represent?
- c) (6 points) If the separate sine and cosine series of part (b) are differentiated term by term, what functions (or generalized functions) do they now represent? Do they converge uniformly on some interval?
2. ( $2 \times 7$  points) Calculate the Fourier transforms of
- a)  $\exp(-a^2 x^2)$  Take  $a > 0$ . (hint: complete the square in the Fourier exponential, use Cauchy's integral theorem, and then the fact that  $\int_{-\infty}^{\infty} \exp(-s^2) ds = \sqrt{\pi}$ )
- b)  $1/x$ . Your answer should be valid for both signs of  $f$ . (hint: treat the Fourier integral as a principal value and use residue theory).
3. Consider the heat/diffusion equation in an infinite medium: for temperature/concentration  $c$  with constant diffusion coefficient  $D$ :

$$D\nabla^2 c = \frac{\partial c}{\partial t} \quad (2)$$

- a) (10 points) Solve for  $c(x, t)$  the 1-dimensional version of eq (2)

$$D \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} \quad (3)$$

with initial condition  $c(x, 0) = c_0 \delta(x - \xi)$  and boundary condition  $c(\pm\infty, t) = 0$ . Solve this equation by Fourier transforming the PDE. When you Fourier transform the PDE you should get an ODE in  $t$  for the Fourier transform  $\tilde{c}(f, t)$ . Fourier transform the initial condition to determine  $\tilde{c}(f, 0)$  at  $t = 0$ , and then inverse transform. You should find that

$$c(x, t) = \frac{c_0}{(4\pi Dt)^{1/2}} \exp -([x - \xi]^2 / 4Dt) . \quad (4)$$

[Hint: one of the results of Problem 2 will be useful, and the Fourier transform has a shift theorem just like the Laplace transform.]

- b) (10 points) Now repeat the calculation for the 3-dimensional version of eq (2), and solve for  $c(x, y, z, t)$  with initial condition:  $c(x, y, z, 0) = c_0\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$ . You should find that

$$c(\vec{x}, t) = \frac{c_0}{(4\pi Dt)^{3/2}} \exp -\left[\frac{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}{4Dt}\right]. \quad (5)$$

- c) (5 points) Show that you can superpose this result (a Green's function) to find the solution to the diffusion equation for arbitrary initial conditions,  $c(x, y, z, 0) = g(x, y, z)$ .

4. ( $5 \times 5$  points) Consider the equation

$$\frac{d^2y}{dx^2} + \lambda^2 y = f(x), \quad y(0) = 0, \quad y'(1) = 0 \quad (6)$$

- a) Find the eigenvalues  $\lambda$  and eigenfunctions for the homogeneous problem  $f(x) = 0$ .  
 b) Find the Green's function (i.e. the solution for  $f(x) = \delta(x - \xi)$ ) when  $\lambda$  is *not* an eigenvalue.  
 c) Use this to find the solution to eq (6), still assuming  $\lambda$  is not an eigenvalue. Is the solution unique? Is there any restriction on  $f(x)$  for the solution to exist?  
 d) Now suppose  $\lambda$  is an eigenvalue. Is there any restriction on  $f(x)$  for a solution to exist? Is the solution then unique?  
 e) Again assuming  $\lambda$  is an eigenvalue and any restrictions on  $f(x)$  are satisfied, give the solution to eq (6) in Green's function form.

The last problem is for extra credit. Part (a) explores further the relation between discrete eigenfunction expansions on finite intervals and the continuous (integral transform) case that arises on infinite intervals. The (independent) part (b) rederives the same result in a totally different way, and offers an opportunity for more practice in solving PDEs by Fourier transforms (like problem 3). Both introduce the remarkable Hankel transform which (in contrast to Fourier and Laplace transforms) has the elegant property that the forward and inverse transforms are identical.

5. ( $2 \times 15$  points extra credit) Consider the axisymmetric drum solved in class (cf. also PS6, #3, but with no  $\phi$  dependence), but now let the radius of the drum go to infinity.

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}; \quad u(r, 0) = f(r), \quad u_t(r, 0) = 0; \quad u(0, t) = \text{finite}, \quad u(\infty, t) = 0. \quad (7)$$

There are many ways to find the solution. Here are two:

- a) Find the solution to this problem by taking the limit of the series solution derived in class (the same as the  $n = 0$  part of PS6, #3) as the radius of the drum  $a \rightarrow \infty$ :

$$u(r, t) = \sum_{m=1}^{\infty} C_m J_0(\alpha_m r/a) \cos(\alpha_m ct/a), \quad \text{where} \quad (8)$$

$$C_m = \frac{\int_0^a r f(r) J_0(\alpha_m r/a) dr}{\int_0^a r J_0^2(\alpha_m r/a) dr}. \quad (9)$$

Here  $\alpha_m$  is the  $m$ th root of  $J_0(x) = 0$ . The procedure is very similar to the procedure followed in class to derive the Fourier transform by letting the period of a complex

Fourier series  $2L \rightarrow \infty$ . In doing this you will need two results (which you need not derive) that can be found using the asymptotic form of the Bessel function for large argument (also derived in class):

$$\alpha_k \rightarrow \pi(k - 1/4) \quad \text{and} \quad (10)$$

$$\int_0^1 (x J_0^2(\alpha_k x) dx = J_1^2(\alpha_k)/2 = J_0'^2(\alpha_k)/2 \rightarrow 1/(k\pi^2) \quad (11)$$

for large  $k$ . As in the Fourier case, you will make use of the fact that an integral of a function is defined as the limit of the sum of the areas of rectangles under the function, as the width of the rectangles goes to zero. You should find

$$u(r, t) = \int_0^\infty \left[ \int_0^\infty s f(s) J_0(\lambda s) ds \right] \lambda J_0(\lambda r) \cos(\lambda ct) d\lambda \quad (12)$$

You have not only solved the PDE, but (if you set  $t = 0$  in this equation) you have also just derived for yourself the Hankel transform of order zero of  $f(s)$  (in square brackets) and its inverse transform. This procedure generalises to the nonaxisymmetric case and gives the general (order  $n$ ) Hankel transform pair.

- b) Solve the differential equation in  $x, y$  cartesian coordinates (the drumhead is infinite after all, so unlike the finite radius case, it doesn't matter whether the boundary condition is applied at a circular infinity or a square one!) by Fourier transforming the PDE, and deriving an ODE in  $t$  for

$$\tilde{u}(f_x, f_y, t) = \int_{-\infty}^\infty \int_{-\infty}^\infty u(x, y, t) \exp(-i2\pi f_x x) \exp(-i2\pi f_y y) dx dy; . \quad (13)$$

Solve the ODE. You should find  $\tilde{u} = A(k) \cos(ckt)$ , where  $k = 2\pi\sqrt{f_x^2 + f_y^2}$ . Do the inverse transform to find  $u(x, y, t)$ . Apply the initial conditions to determine  $A(k)$ . Then convert your 2-D cartesian Fourier integrals to polar coordinates and integrate over the angle (on which nothing depends, by axisymmetry). In doing this, you will want to use the following identity <sup>1</sup>

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \cos \theta) d\theta \quad (14)$$

You should find the same solution eq (12) as you found in part (a).

**Total points: 86**

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<sup>1</sup>If you are wondering where this comes from: you can derive it by Laplace transforming the Bessel differential equation for  $J_0$ , to discover that the Laplace transform of  $\mathcal{L}[J_0] = 1/\sqrt{1+s^2}$ . Then use the Mellin inversion, and collapse the integral to lie along the branch cut.