Problem Set 5

Feb 13, 2004 | Due Feb 20, 2004
ACM 95b/100b | 3pm in Firestone 303
E. Sterl Phinney
(2 pts) Include grading section number

Useful readings: Chapters 6, 8 and 11 of Carrier and Pearson. Arfken Chapter 9 on "Sturm-Liouville Theory and Orthogonal Functions", plus bits of Chapters 11-14 (details of solutions to particular famous S-L problems) if you get stuck somewhere. *In this problem set, you may use any combination of computer algebra software, programming languages and graphics programs you like to do integrals, evaluate coefficients, sum terms and graph them. You may also collaborate with others in learning how to do these things.* But in the end you should still write your own code —modifying code that sums the geometric series and plots y=1/(1-x) to sum and plot what you need is ok; simply printing out a Mathematica notebook with the problem solution is not. One possible starting point for problems 4 and 5, is the Maple notebook with two sample problems at `http://www.its.caltech.edu/~esp/acm95b/ps5examp.mws`. Examples in other languages may be posted later.

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1. (10 points) Consider the linearly independent set $\mathcal{S}$ of functions $\{1, x, x^2, x^3, \ldots\}$ on $-1 \leq x \leq 1$. Define the inner product between real functions f and g to be $\int_{-1}^{1} f(x)g(x)w(x)\,dx$, where the weight $w(x) = 1/\sqrt{1-x^2}$. Use the Gram-Schmidt orthogonalisation procedure to generate from $\mathcal{S}$ the first three members of the set of polynomials orthonormal under that inner product (i.e. your polynomials $T_0(x)$, $T_1(x)$, $T_2(x)$ should be respectively of degree 0, 1, 2). The process is very similar to the example done in class with $w(x) = 1$ which generated the first three (normalised) Legendre polynomials.

2. (5×4 points) If it is possible, put the following eigenvalue problems into Sturm-Liouville form (all constants and functions are to be taken as real-valued), and classify them as defined in class: A: regular Sturm-Liouville, B: Singular Sturm-Liouville or C: Periodic Sturm-Liouville. For the problems that can be put in Sturm-Liouville form, your answers should identify the relevant $p(x)$, $q(x)$, weight function $w(x)$, and the aspects of the boundary conditions which make it a S-L problem.

a) $x^2y'' + xy' + (k^2x^2 - n^2)y = 0$, for $0 \leq x \leq c$, BC: $y(c) = 0$, $y(0)$ finite. k is a constant, n a nonnegative integer.
b) $y'' + k^2y = 0$, for $0 \leq x \leq 2\pi$, BC: $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$. k is a constant. What happens if $k = 0$?
c) $y'' + k^2y = 0$, for $0 \leq x \leq 1$, BC: $y(0) = 0$, $y(1) = 1$. Take $k \neq 0$.
d) $(1-x^2)y'' - 2xy' + \lambda y = 0$, for $-1 \leq x \leq 1$, BC: $y(-1)$, $y(1)$, $y'(-1)$, $y'(1)$ all finite.
e) $(1-x^2)y'' - xy' + \lambda y = 0$, for $-1 \leq x \leq 1$, BC: $y(-1)$, $y(1)$, $y'(-1)$, $y'(1)$ all finite.

3. (3×5 points) The orthogonal polynomials you started to generate in problem 1 are the Chebyshev polynomials, which are eigenfunctions of a second-order Sturm-Liouville problem (whose differential equation, not needed here, is unfortunately seared into our collective memories). The Legendre polynomials are eigenfunctions of another second-order Sturm-Liouville problem. The theorems outlined in class immediately tell you that any square-integrable functions can be expanded in terms of them, and that the expansion coefficients must obey Parseval's Theorem.

a) Illustrate this by finding the coefficients of the expansion of the function $f(x) = 1 - x^2$ in terms of

i) The first three orthogonal Legendre polynomials. Why would the coefficients of subsequent Legendre polynomials in the expansion of $f(x)$ be zero?

ii) The first three orthogonal Chebyshev polynomials. Why would the coefficients of subsequent Chebyshev polynomials in the expansion of $f(x)$ be zero?

b) Verify that Parseval's Theorem is satisfied by your answers to the previous part. If you use the normalized form of Parseval's theorem, be careful to check that your Legendre and Chebyshev polynomials are properly normalized.

4. (5×6 points) The equation of motion for small transverse displacements $y(x)$ of a flexible string of length L, uniform mass per unit length μ and constant tension T subject to an external force per unit length $f(x)$ is

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} + f(x), \quad y(0) = y(L) = 0 \tag{1}$$

In this problem set we'll do steady-state, where the left-hand side is zero, and the equation reduces to an ODE. Next problem set will put the time dependence back in and design a (crude) piano.

a) Find the equilibrium shape $y(x)$ of a string being pulled (plucked) at $x = \xi$ $(0 < \xi < L)$, i.e. solve for $y(x)$ when $f(x) = F_0 \delta(x - \xi)$, where F_0 is the applied force.

b) Use your Green's function from (a) to find the shape of the string sagging under the force of gravity $f(x) = \mu g$, where g is the usual gravitational acceleration on earth.

c) Find the eigenfunctions and eigenvalues of the homogeneous $d^2y/dx^2 + \lambda y = 0$ Sturm-Liouville problem, with the boundary conditions on y as given in the statement of the problem. Make the eigenfunctions orthonormal.

d) Set $L = 1$, $\xi = 1/7$ and $F_0/T = 1$ (pretend the equation is still valid —this just makes a nice normalisation of the linear problem!). Expand your solution to part (a) in terms of the eigenfunctions you found in part (c). Give a formula for the coefficients in the expansion, and evaluate the first 9 of them numerically to 4 significant figures. How close does the sum of squares of just those 9 coefficients come to satisfying Parseval's theorem? [optional: if $\xi = 1/n$ for integer n (here $n = 7$) is there something special about the nth, $2n$th, $3n$th etc coefficients?]

e) Sum the first 3, 10, 30 and 100 terms of your answer to (d) and graph each of the 4 partial sums as a function of x. Compare them graphically to your solution in (b) by both overplotting the functions, and by graphing the residuals (partial sum -exact solution) as a function of x. How does the maximum error scale with the number of terms in the partial sum?

5. (2×10 points) Repeat parts (d) and (e) of the previous problem for the eigenfunction expansions of the following functions defined on $0 \le x \le 1$:

a) $y(x) = 1$ for $1/4 < x < 3/4$, $y(x) = 0$ otherwise. Be careful that your density of plotting points is high enough to resolve all the features when you get up to 100 terms. Does it look like the eigenfunction expansion is uniformly converging of to the specified $y(x)$?

b) $y(x) = 1/\sin(\pi x)$. What is going on here? Are we violating any of the conditions of theorems about convergence in the mean for expansions of functions in terms of Sturm-Liouville eigenfunctions?

Total points: 97