Problem Set 1, REVISED

January 9, 2004 — Due January 16, 2004
ACM 95b/100b — 3pm in Firestone 303
E. Sterl Phinney — (2 pts) Include grading section number[1]

Note new collaboration policy:
While working on the problem sets, you should first try to work each problem **alone** consulting only the lecture notes and the textbooks. If after about 30 minutes you still feel stuck, you may consult other books and people (friends, TAs etc) enough to get you unstuck. But you should write your solution on your own (not while looking at or listening to someone else's solution or solution outline), so that you actually think through it. After you have what you think is a solution, you may check your final answer in any way you see fit. If you find it is wrong, however, you should rework the problem as described above. Failing to learn to think for yourself by slavishly copying solutions from other people, books, previous years' solution sets, etc is (a) a waste of your time, and (b) an honor code violation for the purposes of this course.

---

1. (2×7 points) Solve the following linear initial value problems and in each case describe the interval on which the solution is defined

a) $y' + 2xy = e^{-x^2}$ with $y(0) = -1$

b) $ty' + 2y = t^2 - t + 1$ with $y(1) = 1/2$

2. (15 points) Solve the following real-valued initial value problem [4 points]:

$$xy'(x) + Ay(x) = 1 + x^2 \text{ with } y(1) = 1\,, \tag{1}$$

for all (positive, zero and negative) values of the constant A, and then answer the following questions: Is your solution is actually valid for all values of the constant A? If you weren't careful in your initial solution, you many need to amend it to include some different functional forms for some values of A. [4 points]. Over what range of x is the solution defined and continuous; your answer may depend on A [3 points]? Ignore the condition $y(1) = 1$ and find that function $y(x)$ which obeys the differential equation and is bounded at the origin; you need not find this solution for the special values of A that require different functional forms [4 points].

3. (10 points) Consider the same DE as in problem 2 complexified, i.e. with real x replaced by complex z and real $y(x)$ replaced by complex $w(z)$. Assume $w(1 + 0i) = 1 + 0i$. Find an analytic solution $w(z)$ for general real A (you may ignore any 'peculiar values' of A for purposes of this problem), and discuss the region of the complex plane over which it is valid (5 points). Be sure to define the locations of any branch cuts you introduce (2 points). Under what circumstances can you analytically continue the solution over the whole real line $z = x + 0i$? To the whole real line excluding one point? Compare to your answers to your answer for problem 2 (3 points).

---

[1] Please continue to use the same grading section number as last term. If you do not already have a grading section number, go to the offical ACM95b/100b web site:
`http://www.its.caltech.edu/~esp/acm95b/ACM95b.html`
and follow the link to the ACM95b/100b Underground. Create a login account and include the assigned grading section number at the beginning of every problem set and exam. You will automagically be entered into a database that will send you emails about problem set corrections, course changes and the like. The system will also enable you to keep track of your grades, ask questions of TAs, whine, etc.

4. (15 points) For any real number a, find $y(x)$ such that

$$y'(x) + ay(x) = e^{-x} \quad \text{with} \quad y(0) = 1\,, \tag{2}$$

over the range $0 \le x < \infty$ [8 points]. Sketch a graph of $y(x)$ over that range for each of several values of a. Include particularly, $a = -100, -1, 1, 100, 10^4$ [7 points]

5. (7 points) Consider the DE

$$y' + y/x = 1 \quad \text{with} \quad y(1) = 1/2\,. \tag{3}$$

Find the vertices of the Euler polygonal approximations (described in class) to the solution, as a function of x-increment h [5 points]. Show explicitly that the approximate Euler solutions approach the actual solution as $h \to 0$ [2 points].

6. (4×5 points)
a) A cylindrical bucket has a hole in the bottom, and is observed to be empty at time $\tau = 0$. The differential equation governing the height $z(\tau)$ of water in the bucket as a function of (appropriately scaled) time is the following 'final value problem':

$$dz/d\tau = -|z|^{1/2} \quad \text{with} \quad z(0) = 0\,, \tag{4}$$

Notice the following: (i) height of water is a positive number (on my rulers anyway), so only solutions with $z \ge 0$ are to be considered, (ii) $z(\tau) = 0$ is *a* solution, but perhaps not the only one. Given that $z(0) = 0$, show mathematically that the solution is unique for $\tau > 0$, but nonunique for $\tau < 0$, and give a simple explanation in terms of what you can infer (when did it empty?) from observing an empty bucket with a puddle under it for why nonuniqueness is reasonable and physical. [hint: be careful with integration constants and matching solutions]

With your insights from part (a), now consider the following initial value problem:

$$dz/dt = |z|^{p/q} \quad \text{with} \quad z(0) = 0\,, \tag{5}$$

where p and q are positive integers with no common factors. Notice that $z(t) = 0$ is *a* solution, but perhaps not the only one, and that 'initial value problem' means that you are to consider only $t \ge 0$. Notice also that setting $\tau = -t$ in part (a) converts that 'final value problem' to the present initial value problem, with $p = 1, q = 2$.

b) Show that there are an infinite number of solutions if $p < q$.

c) Show that there is a unique solution if $p > q$.

d) Relate your results of parts (b) and (c) to the Lipschitz condition used in the proof of the uniqueness theorem sketched in class.

7. (5×4 points) The object of this wordy problem is to give you practice in finding exact solutions to one common type of ODE, and also show you that approximate solutions are often much more useful than exact solutions if you want to *understand* what is going on.

If the Mars Exploration Rover folks at JPL had taken only one week of ACM95b, they might unwisely have decided to send the Rovers down radially (i.e. vertically into Mars' atmosphere) because they did not yet know how to solve the systems of coupled nonlinear differential equations that would govern an oblique approach. Consider this simplified radial problem. Let z denote height above Mars' surface, and $\rho = \rho_0 \exp[-z/H]$ the density of its atmosphere, whose surface density is $\rho_0 = 2 \times 10^{-5}$g cm^{-3} and whose scale height is $H = 11$km. Let the Rover (with heat shield!) have constant mass m and effective area A_e (note for cognoscenti, unimportant for this problem: $A_e = Ac_D/2$, where A is the actual cross-sectional area, and $c_D \sim 2$ is the drag coefficient). The drag force on a Rover moving straight down through the atmosphere at speed $v = -dz/dt$ is $F_D = \rho v^2 A_e$. Also acting on the Rover is the force of Mars' gravity ($g(0) = 373$cm s^{-2} at the surface, with vertically integrated escape velocity $v_e(0) = [2\int_0^\infty g(z)dz]^{1/2}$ from Mars of 5km s^{-1}).

a) Show that the Rover's equation of motion is

$$-m\frac{dv}{dt} = -mg(z) + v^2 \rho_0 A_e e^{-z/H} \qquad (6)$$

Show also that the left hand side of eq 6 can be written as $mv(dv/dz) = md(v^2/2)/dz$, thus changing the independent variable from time t to height z above the surface of Mars. Simplify the equation by noting that only the combined quantity $m/(\rho_0 A_e) \equiv l_s$ (the 'stopping length in the Martian surface atmosphere') appears. Notice that this is a first order linear ODE in v^2.

b) First pretend $g(z) = 0$. Solve the resulting homogeneous equation for $v^2(z)$, given a speed of incidence on the atmosphere $v(\infty) = v_\infty$. Your expression for $v(z)$ should involve only (elementary functions and integrals of) H, l_s, v_∞ and z.

c) Now allow a $g(z) \neq 0$. Use your result in (b) to find the general solution for $v^2(z)$, given a speed of incidence $v(\infty) = v_\infty$. Your expression for $v(z)$ should involve only (elementary functions and integrals of) $g(z)$, H, l_s, v_∞ and z.

d) Show that the complicated expression you found in (c) has two simple limiting cases, $H/l_s \ll 1$ and $H/l_s \gg 1$. For each of these two limits, find the lowest order approximation expressions for the Rover landing velocity $v(z = 0)$. Also explain how you could have derived these two limiting answers immediately by inspection of the differential equation you found in part (a).

e) To avoid catastrophic destruction of the Rover, you should find from (d) that you want to be in the $H/l_s \gg 1$ limit. What is the maximum radius (in centimeters) a spherical Rover of mean density 1g cm^{-3} could have if it is to slow down to $v(0) < 10$m s^{-1}? Take $v_\infty = 5$km s^{-1}. If the answer makes you worried, you now understand why the actual Rover came in obliquely, and had a parachute and retro rockets.

Total points: 103