Midterm Exam Solutions

Problem 1 (8+3×6 Points)

(a) (8 points)

(i) (2 points)

$$y'' + x\,y' + \frac{2}{x^3}\,y = 0 \tag{1}$$

Irregular (Essential) Singular point at x=0.

(ii) (2 points)

$$y'' - \frac{x}{(x-1)(x-2)}\,y' + \frac{x^2}{(x-1)(x-2)}\,y = 0 \tag{2}$$

Regular singular points at x=1 and x=2. (1 point each)

(iii) (2 points)

$$y'' + 2\,y' + 2\,y = 0 \tag{3}$$

No singular points, all ordinary points.

(iv) (2 points)

$$y'' + \frac{x}{e^x - 1}\,y = 0 \tag{4}$$

Regular Singular points occur for $x=2n\pi i$ (1 point) except for x=0 which is an ordinary point since the singularity is removable(1 point):

$$\lim_{x\to 0} \frac{x}{e^x - 1} = 1 \tag{5}$$

(b) (6 points)

(i) (3 points)

Since there are singular points at x=1 and x=2, the nearest singular point to the origin is x=1. (1 point) By Fuch's theorem, the radius of convergence of the series is at least the distance to the nearest singularity, so the radius of convergence is at least 1 (1 point). Hence the series is guaranteed to converge in (1 points)

$$|x| < 1 \tag{6}$$

(ii) (3 points)

Since there are singular points at $x=2n\pi i$, the nearest singular points to the origin are $x=\pm 2\pi i$ (1 points). By Fuch's theorem, the radius of convergence of the series is at least the distance to the nearest singularity, so the radius of convergence is at least 2π (1 point). Hence the series is guaranteed to converge in (1 point)

$$|x| < 2\,\pi \tag{7}$$

(c) (6 points)

(i) (2 points)

As stated in class, an equation of the form

$$y'' + \frac{p(x)}{(x - x_0)} y' + \frac{q(x)}{(x - x_0)^2} y = 0 \tag{8}$$

Where p and q are analytic at x_0 leads to an indicial equation given by

$$\nu^2 + (p_0 - 1)\nu + q_0 = 0 \tag{9}$$

Where

$$\begin{aligned} p(x) &= \sum_{n=0}^{\infty} p_n (x - x_0)^n \\ q(x) &= \sum_{n=0}^{\infty} q_n (x - x_0)^n \end{aligned} \tag{10}$$

In this problem the equation is

$$y'' - \frac{x}{(x-1)(x-2)} y' + \frac{x^2}{(x-1)(x-2)} y = 0 \tag{11}$$

so

$$\begin{aligned} p &= \frac{-x}{x-2} \\ q &= \frac{x^2 (x-1)}{x-2} \end{aligned} \tag{12}$$

This gives p_0=1 and q_0=0. Plugging these in gives the indicial equation (1 point)

$$\nu^2 = 0 \tag{13}$$

So there is a double root of ν=0 (1 point).

(ii) (2 points)

No, it can not. By Fuch's theorem, only one of the solutions will be in the form of a Taylor series.

(iii) (2 points)

Method 1

Fuch's theorem (or they might site theorem 25 from page 27 of the class notes) tells us that one solution will be of the form of a power series

$$y_1 = \sum_{n=0}^{\infty} a_n (x-1)^n \tag{14}$$

and a second linearly independent solution will be of the form

$$y_2 = \ln|x-1| \, y_1(x) + \sum_{n=0}^{\infty} b_n (x-1)^n \tag{15}$$

Method 2

As $\nu_2 \to \nu_1$ terms of the form

$$x^{n+\nu_1} - x^{n+\nu_2} \tag{16}$$

take the form

$$x^{n+\nu_1} - x^{n+\nu_2} = x^{n+\nu_1}(1 - x^{\nu_2 - \nu_1}) = x^{n+\nu_1}(1 - e^{(\nu_2 - \nu_1)\ln|x|}) \to -x^{n+\nu_1}(\nu_2 - \nu_1)\ln|x| \tag{17}$$

So we expect a solution of the form

$$y_2 = \ln|x-1|\, y_1(x) + \sum_{n=0}^{\infty} b_n (x-1)^n \tag{18}$$

(d) (6 points)

$$\begin{aligned} &y'' + 2y' + 2y = 0 \\ &y(0) = 1 \\ &y'(0) = 0 \end{aligned} \tag{19}$$

Transform

$$s^2 Y - s\, y(0) - y'(0) + 2 s Y - 2 y(0) + 2 Y = 0 \tag{20}$$

Insert the initial data and solve for Y (2 points)

$$Y = \frac{s+2}{s^2 + 2s + 2} \tag{21}$$

Method 1

Rewrite the transform as (2 points)

$$Y = \frac{s+1}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1} \tag{22}$$

From a table we know

$$\begin{aligned} &\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \text{Cos}\, t \\ &\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \text{Sin}\, t \end{aligned} \tag{23}$$

Using these together with a shifting theorem gives (2 points)

$$y = e^{-t} (\text{Sin}\, t + \text{Cos}\, t) \tag{24}$$

Method 2

Rewrite the transform as partial fractions (2 points)

$$Y = \frac{\frac{1-i}{2}}{s - (-1+i)} + \frac{\frac{1-i}{2}}{s - (-1-i)} \tag{25}$$

From a table

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 \tag{26}$$

Applying a shifting theorem gives (2 points)

$$y = \frac{1-i}{2} e^{(-1+i)t} + \frac{1+i}{2} e^{(-1-i)t} = e^{-t} (\text{Sin}\, t + \text{Cos}\, t) \tag{27}$$

Method 3

$$y = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s+2}{s^2+2s+2} e^{st}\, ds \tag{28}$$

Since there are simple poles at s=-1±i, we choose and c≥0. We then integrate around the following contour, call it Γ_R.

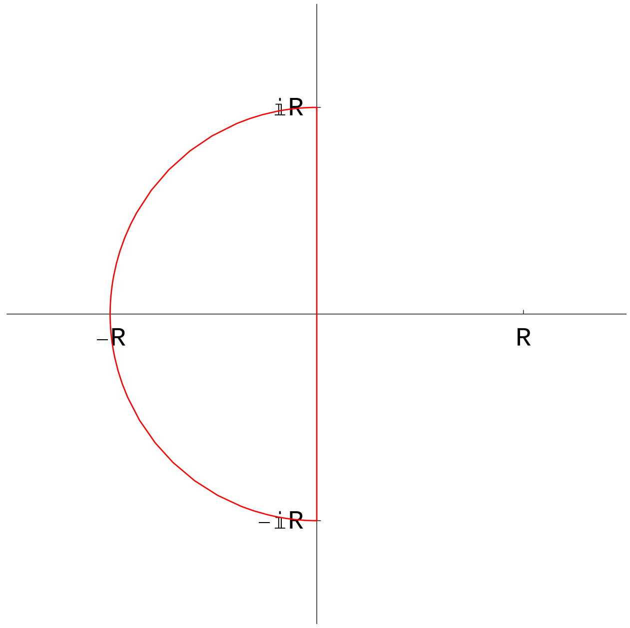

Figure 1

The integral around the semicircular contour vanishes which is proven as follows (2 points)

$$\left|\int_{C_R} \frac{s+2}{s^2+2s+2} e^{st}\, ds\right| = \left|\int_{\pi/2}^{3\pi/2} \frac{R e^{i\theta}+2}{(R e^{i\theta})^2+2R e^{i\theta}+2} e^{R e^{i\theta} t} R\, i\, e^{i\theta}\, d\theta\right| \leq R \int_{\pi/2}^{3\pi/2} \frac{|R e^{i\theta}+2|}{|R^2 e^{2i\theta}+2R e^{i\theta}+2|} e^{R t \operatorname{Cos}\theta}\, d\theta \tag{29}$$

For Large R we have by the triangle inequality

$$\frac{|R e^{i\theta}+2|}{|R^2 e^{2i\theta}+2R e^{i\theta}+2|} \leq \frac{|R e^{i\theta}|+|2|}{|R^2 e^{2i\theta}|-|2R e^{i\theta}|-|2|} = \frac{R+2}{R^2-2R-2} \to 0 \tag{30}$$

Since Cosθ is non-positive on the interval (π/2,3π/2) Jordan's Lemma applies and the integral vanishes. So, by the residue theorem we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{s+2}{s^2+2s+2} e^{st}\, ds = \operatorname{Res}\left(\frac{s+2}{s^2+2s+2} e^{st}, -1-i\right) + \operatorname{Res}\left(\frac{s+2}{s^2+2s+2} e^{st}, -1+i\right) \tag{31}$$

Calculating the residues (1 point)

$$\begin{aligned}
\lim_{s\to -1-i} \frac{(s-(-1-i))(s+2)}{s^2+2s+2} e^{st} &= \lim_{s\to -1-i} \frac{(s+2)e^{st}}{s-(-1+i)} = \frac{1+i}{2} e^{(-1-i)t} \\
\lim_{s\to -1+i} \frac{(s-(-1+i))(s+2)}{s^2+2s+2} e^{st} &= \lim_{s\to -1+i} \frac{(s+2)e^{st}}{s-(-1-i)} = \frac{1-i}{2} e^{(-1+i)t}
\end{aligned} \tag{32}$$

So we have (1 point)

$$y = \frac{1+i}{2} e^{(-1-i)t} + \frac{1-i}{2} e^{(-1+i)t} = e^{-t}(\operatorname{Sin} t + \operatorname{Cos} t) \tag{33}$$

Problem 2 (4×6 points)

(a) (6 points)

As given in class, the Wronskian of two solutions to an equation of this form

$$p\, y'' + q\, y' + r\, y = 0 \tag{34}$$

is of the form (2 points)

$$W(x) = C\, e^{-\int \frac{q(x)}{p(x)}\, dx} \tag{35}$$

For this example q(x)=0, so W(x)=C, a constant. From the definition of the Wronskian, and the given initial conditions, we calculate W(0) (2 points)

$$W(0) = y_1(0)\, y_2'(0) - y_1'(0)\, y_2(0) = 1 \tag{36}$$

Since W(x)=C, a constant, and W(0)=1 we have learned that W(x)=1 for all x (2 points).

(b) (6 points)

Since W(x)=1≠0 for all x, these two solutions are linearly independent (3 points). Since a second order linear homogeneous ODE has a general solution composed of a linear combination of any two linearly independent solutions, we know that the general solution to the ODE is (3 points)

$$y(x) = A\, y_1(x) + B\, y_2(x) \tag{37}$$

for arbitrary constants A and B.

(c) (6 points)

$$y = \sum_{n=0}^{\infty} a_n\, x^n \tag{38}$$

Plug into the ODE

$$\sum_{n=2}^{\infty} a_n\, n\, (n-1)\, x^{n-2} + \sum_{n=0}^{\infty} 2\, a_n\, x^n + \sum_{n=0}^{\infty} -4\, a_n\, x^{n+2} = 0 \tag{39}$$

Shift the indices of the first and last sum

$$\sum_{n=0}^{\infty} a_{n+2}\, (n+2)\, (n+1)\, x^n + \sum_{n=0}^{\infty} 2\, a_n\, x^n + \sum_{n=2}^{\infty} -4\, a_{n-2}\, x^n = 0 \tag{40}$$

Combine these into a single sum (2 points)

$$2\, (a_2 + a_0) + 2\, (3\, a_3 + a_1)\, x + \sum_{n=2}^{\infty} (a_{n+2}\, (n+2)\, (n+1) + 2\, a_n - 4\, a_{n-2})\, x^n = 0 \tag{41}$$

The only power series that is zero throughout its radius of convergence is the series with all zero coefficients. Setting the coefficients to zero gives (1 point)

$$\begin{aligned} a_2 &= -a_0 \\ a_3 &= -a_1 / 3 \\ a_{n+2} &= \frac{4\, a_{n-2} - 2\, a_n}{(n+2)\, (n+1)} \quad \text{for } n \geq 2 \end{aligned} \tag{42}$$

We calculate the first few terms (2 points)

$$a_0 \left(1 - x^2 + \frac{1}{2}\, x^4 - \ldots\right) + a_1 \left(x - \frac{1}{3}\, x^3 + \frac{7}{30}\, x^5 - \ldots\right) \tag{43}$$

To satisfy the boundary conditions we define (1 point)

$$y_1(x) = 1 - x^2 + \frac{1}{2} x^4 - \ldots$$
$$y_2(x) = x - \frac{1}{3} x^3 + \frac{7}{30} x^5 - \ldots \quad (44$$

(d) (6 points)

Method 1

Since one solution of the equation is

$$y_1(x) = e^{-x^2} \quad (45)$$

we seek a second solution of the form (1 point)

$$y_2(x) = e^{-x^2} u(x) \quad (46)$$

Plug this into the ODE

$$\left(e^{-x^2} u\right)'' + (2 - 4x^2) e^{-x^2} u = 0 \quad (47)$$

Compute the derivatives and simplify (1 point)

$$u'' - 4 x u' = 0 \quad (48)$$

Using an integrating factor, or looking up the general solution formula for first order ODE leads to (1 point)

$$u' = A e^{2x^2} \quad (49)$$

Integrating this once gives (1 point)

$$u = B + A \int_0^x e^{2t^2} \, dt \quad (50)$$

and (1 point)

$$y_2(x) = e^{-x^2} \left(B + A \int_0^x e^{2t^2} \, dt\right) \quad (51)$$

Fitting the initial data gives two equations for A and B

$$0 = y_2(0) = e^{-0} \left(B + A \int_0^0 e^{2t^2} \, dt\right) = B$$
$$1 = y_2'(0) = 0 \, e^{-0^2} \left(B + A \int_0^0 e^{2t^2} \, dt\right) + A = A \quad (52)$$

So we conclude (1 point)

$$y_2(x) = e^{-x^2} \int_0^x e^{2t^2} \, dt \quad (53)$$

Method 2

In part A we computed the Wronskian W(x)=1. This gives (1 point)

$$y_1 y_2' - y_1' y_2 = 1 \quad (54)$$

Plugging in y_1 gives a first order ODE for y_2 (1 point)

$$y_2' + 2 x y_2 = e^{x^2} \quad (55)$$

Solving this with an integrating factor or the general solution formula for a first order linear ODE gives (2 point)

$$y_2 = e^{-x^2}\left(B + A\int_0^x e^{2t^2}\, dt\right) \tag{56}$$

Fitting the initial data gives two equations for A and B

$$\begin{aligned} 0 &= y_2(0) = e^{-0}\left(B + A\int_0^0 e^{2t^2}\, dt\right) = B \\ 1 &= y_2'(0) = 0\, e^{-0^2}\left(B + A\int_0^0 e^{2t^2}\, dt\right) + A = A \end{aligned} \tag{57}$$

So we conclude (2 points)

$$y_2(x) = e^{-x^2}\int_0^x e^{2t^2}\, dt \tag{58}$$

Problem 3 (4×6 points)

(a) (6 points)

$$\begin{aligned} &y' - \frac{2}{x}\, y = x^5 \\ &y(1) = 0 \end{aligned} \tag{59}$$

From the class notes, an integrating factor is given by (1 point)

$$I = e^{\int -\frac{2}{x}\, dx} = e^{-2\ln|x|} = x^{-2} \tag{60}$$

This gives (1 point)

$$(x^{-2}\, y)' = x^3 \tag{61}$$

Integrating both sides gives (2 points)

$$y = x^2\left(A + \int_1^x t^3\, dt\right) = A\, x^2 + \frac{1}{4}(x^6 - x^2) \tag{62}$$

Fitting the initial date gives

$$0 = y(1) = A \tag{63}$$

So the solution to the IVP is (2 points)

$$y = \frac{x^6 - x^2}{4} \tag{64}$$

(b) (6 points)

Method 1

$$\begin{aligned} &y_G' - \frac{2}{x}\, y_G = \delta(x - \xi) \\ &y_G(1) = 0 \end{aligned} \tag{65}$$

Using the same integrating factor we found in the previous part gives (2 points)

$$y_G = x^2\left(A + \int_1^x t^{-2}\, \delta(t - \xi)\, dt\right) \tag{66}$$

Fitting the intial data gives A=0. Computing the integral we find (4 points)

$$y_G = \left(\frac{x}{\xi}\right)^2 H(x-\xi)\,H(\xi-1) = \begin{cases} \left(\frac{x}{\xi}\right)^2 & x > \xi > 1 \\ 0 & \text{otherwise} \end{cases} \tag{67}$$

Method 2

The general solution to the homogeneous equation is (1 point)

$$y = A\,x^2 \tag{68}$$

So we have (1 point)

$$y = \begin{cases} A\,x^2 & \text{for } 1 < x < \xi \\ B\,x^2 & \text{for } x > \xi > 1 \end{cases} \tag{69}$$

The initial data gives A=0. Integrating the equation from ξ-ϵ to ξ-ϵ and letting $\epsilon \to 0$ gives (1 point)

$$1 = \int_{\xi^-}^{\xi^+} \delta(x-\xi)\,dx = \int_{\xi^-}^{\xi^+} y_G\,' - \frac{2}{x}\,y_G\,dx = y_G(\xi^+) - y_G(\xi^-) = y_G(\xi^+) \tag{70}$$

This gives B=1/ξ^2(1 point). So our solution is (2 points)

$$y_G = \begin{cases} \left(\frac{x}{\xi}\right)^2 & x > \xi > 1 \\ 0 & \text{otherwise} \end{cases} = \left(\frac{x}{\xi}\right)^2 H(x-\xi)\,H(\xi-1) \tag{71}$$

(c) (6 points)

$$\begin{aligned} & y' - \frac{2}{x}\,y = f(x) \\ & y(1) = 0 \end{aligned} \tag{72}$$

Method 1

By the principle of superposition, we suspect that a solution to this IVP will be (2 points)

$$y = \int_1^x f(\xi)\,y_G(x \mid \xi)\,d\xi = \int_1^x f(\xi)\left(\frac{x}{\xi}\right)^2 H(x-\xi)\,H(\xi-1)\,d\xi = \int_1^x f(\xi)\left(\frac{x}{\xi}\right)^2 d\xi \tag{73}$$

We check this by plugging it into the ODE (2 points)

$$\left(\int_1^x f(\xi)\left(\frac{x}{\xi}\right)^2 d\xi\right)' - \frac{2}{x}\int_1^x f(\xi)\left(\frac{x}{\xi}\right)^2 d\xi = f(x) + \int_1^x f(\xi)\left(\frac{2\,x}{\xi^2}\right) d\xi - \frac{2}{x}\int_1^x f(\xi)\left(\frac{x}{\xi}\right)^2 d\xi = f(x) \tag{74}$$

So this is indeed the solution. Setting f(x)= x^5 gives (2 points)

$$y = \int_1^x \xi^5 \left(\frac{x}{\xi}\right)^2 d\xi = x^2 \int_1^x \xi^3\,d\xi = \frac{x^6 - x^4}{4} \tag{75}$$

Method 2

If we couldn't do part (b), then we first find a solution to the homogeneous problem

$$y' - \frac{2}{x}\,y = 0 \tag{76}$$

Using the same integrating factor we found in part (a) lets us write

$$(x^{-2}\,y)' = 0 \tag{77}$$

Integrating and simplifying gives (1 point)

$$y = A\,x^2 \tag{78}$$

To solve the inhomogeneous problem by variation of parameters we look for a solution of the form (1 point)

$$y(x) = x^2 u(x) \tag{79}$$

Plug this into the ODE

$$(x^2 u)' - \frac{2}{x} x^2 u = f(x) \tag{80}$$

Differentiate and simplify

$$x^2 u' = f(x) \tag{81}$$

Solve for u' and integrate

$$u = A + \int_1^x \frac{f(t)}{t^2} \, dt \tag{82}$$

So the general solution is of the form (1 point)

$$y(x) = x^2 u(x) = A x^2 + x^2 \int_1^x \frac{f(t)}{t^2} \, dt \tag{83}$$

The initial condition gives:

$$0 = y(1) = A \tag{84}$$

So the solution to the inhomogeneous IVP is (1 point)

$$y = x^2 \int_1^x \frac{f(t)}{t^2} \, dt \tag{85}$$

Setting f(t)= t^5 gives (2 points)

$$y = x^2 \int_1^x \frac{t^5}{t^2} \, dt = x^2 \int_1^x t^3 \, dt = \frac{x^6 - x^4}{4} \tag{86}$$

(d) (6 points)

$$\begin{aligned} & y' - \frac{2}{x} y = x^5 \\ & y(0) = 0 \end{aligned} \tag{87}$$

Using the integration factor from part (a) gives

$$(x^{-2} y)' = x^3 \tag{88}$$

Integrating gives (3 points)

$$y = x^2 \left(A + \int_0^x t^3 \, dt \right) = A x^2 + \frac{1}{4} x^6 \tag{89}$$

The initial data is automatically satisfied for any value of A

$$y(0) = A\, 0^2 + \frac{1}{4} 0^6 = 0 \tag{90}$$

So there are infinitely many solutions, one for each choice of the arbitrary constant A(3 points).

Problem 4 (4×6 points)

(a) (6 points)

For a>0 we have (1 point)

$$\mathcal{L}[\delta(t-a)] = \int_0^{\infty} e^{-st}\,\delta(t-a)\,dt = e^{-as} \tag{91}$$

Transforming the series term by term gives (2 points)

$$\mathcal{L}\Big[\sum_{n=0}^{\infty}(-1)^n\,\delta(t-(2n+1)k)\Big] = \sum_{n=0}^{\infty}(-1)^n\,e^{-(2n+1)ks} \tag{92}$$

The sum of a geometric series is given by

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \tag{93}$$

For all |z|<1. Using this we find (3 points)

$$\sum_{n=0}^{\infty}(-1)^n\,e^{-(2n+1)ks} = e^{-ks}\sum_{n=0}^{\infty}\left(-e^{-2ks}\right)^n = \frac{e^{-ks}}{1+e^{-2ks}} = \frac{1}{2}\,\frac{1}{\frac{e^{ks}+e^{-ks}}{2}} = \frac{1}{2\,\mathrm{Cosh}(ks)} \tag{94}$$

(b) (6 points)

Method 1

From the table of transforms given in class (3 points)

$$\mathcal{L}^{-1}(G) = \mathcal{L}^{-1}\left(\frac{F}{s}\right) = \int_0^t f(x)\,dx \tag{95}$$

Method 2

Consider differentiating the inverse Laplace transform

$$\frac{d}{dt}\,\mathcal{L}^{-1}\left(\frac{F}{s}\right) = \frac{d}{dt}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(s)}{s}\,e^{st}\,ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)\,e^{st}\,ds = f(t) \tag{96}$$

Integrating this gives for some constant A (1 point)

$$g = \mathcal{L}^{-1}(G) = \mathcal{L}^{-1}\left(\frac{F}{s}\right) = A + \int_0^t f(x)\,dx \tag{97}$$

Notice that g'=f and that g(0)=A. Using the formula for the transform of a derivative (1 point)

$$F = \mathcal{L}[f] = \mathcal{L}[g'] = s\mathcal{L}[g] - g(0) = s\,G - A \tag{98}$$

This gives:

$$g = \mathcal{L}^{-1}(G) = \mathcal{L}^{-1}\left(\frac{F}{s}\right) = \mathcal{L}^{-1}\left(G - \frac{A}{s}\right) = g - A \tag{99}$$

So we must have A=0. Hence (1 points)

$$\mathcal{L}^{-1}\left(\frac{F}{s}\right) = \int_0^t f(x)\,dx \tag{100}$$

Regardless of which method is used, employing the function in part (a) we find (1 point):

$$\mathcal{L}^{-1}\left(\frac{F}{s}\right) = \int_0^t f(x)\,dx = \int_0^t \sum_{n=0}^{\infty} (-1)^n \delta(t-(2n+1)k)\,dx = \sum_{n=0}^{\infty}(-1)^n H(t-(2n+1)k) = \begin{matrix} 1 & (4n+1)k \le t < (4n+3)k \\ 0 & (4n+3)k \le t < (4n+5)k \end{matrix} \tag{101}$$

Plotting (2 points)

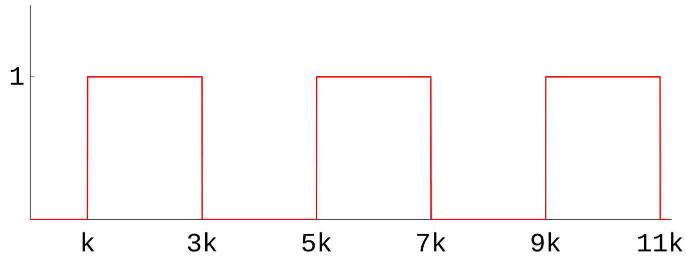

Figure 2

(c) (6 points)

$$\mathcal{L}^{-1}\left[\frac{1}{2s\,\mathrm{Cosh}(k s)}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{2s\,\mathrm{Cosh}(k s)}\,ds \tag{102}$$

The roots of the denominator are s=0, and the roots of

$$e^{ks} + e^{-ks} \tag{103}$$

Letting $s=a+bi$, simplifying and setting the real and imaginary parts to zero gives

$$\begin{aligned} &\mathrm{Cos}(b k)\,\mathrm{Cosh}(a k) = 0 \\ &\mathrm{Sin}(b k)\,\mathrm{Sinh}(a k) = 0 \end{aligned} \tag{104}$$

Since Cosh has no real roots, the first equation is only satisfied for $bk=(2n+1)\pi/2$. The second equation then gives Sinh(ak)=0. The only real root of Sinh is 0, so a=0. Hence the roots of the denominator are s=0 and (1 point)

$$s_n = \frac{(2n+1)\pi}{2k}\,i \quad \text{for } n = .., -2, -1, 0, 1, 2, \;.. \tag{105}$$

These, together with s=0, are all on the imaginary s axis. So we can choose any c>0. Use the following contour

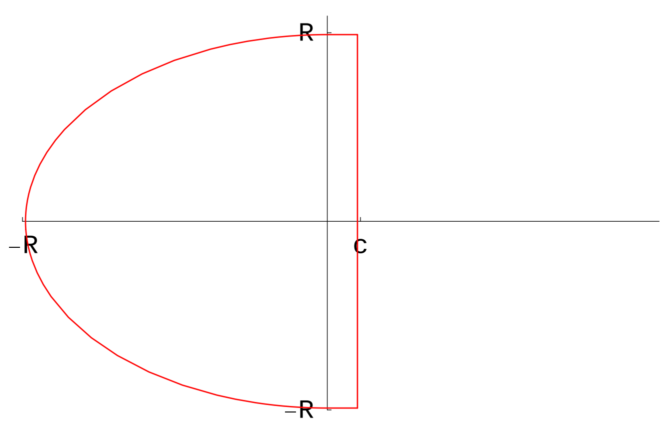

Figure 3

This contour, call it Γ_R is composed of the Bromwich contour, a semicircular piece, and two horizontal parts

$$\int_{\Gamma_R} \frac{e^{st}}{2\,s\,\text{Cosh}(k\,s)}\,ds =$$

$$\int_{c-iR}^{c+iR} \frac{e^{st}}{2\,s\,\text{Cosh}(k\,s)}\,ds + \int_{C_R} \frac{e^{st}}{2\,s\,\text{Cosh}(k\,s)}\,ds + \int_{c+iR}^{iR} \frac{e^{st}}{2\,s\,\text{Cosh}(k\,s)}\,ds + \int_{-iR}^{c-iR} \frac{e^{st}}{2\,s\,\text{Cosh}(k\,s)}\,ds \qquad (106)$$

We first show that the integrals on the horizontal portions vanish (1 point)

$$\left|\int_{c+iR}^{iR} \frac{e^{s\,t}}{2\,s\,\text{Cosh}(k\,s)}\,ds\right| =$$

$$\frac{1}{2}\left|\int_{c}^{0} \frac{e^{(x+iR)\,t}}{(x+i\,R)\,\text{Cosh}(k\,(x+i\,R))}\,dx\right| \le \frac{1}{2}\int_0^c \frac{e^{x\,t}}{|x+i\,R|\,|\,\text{Cosh}(k\,x+i\,k\,R)|}\,dx \qquad (107)$$

A simple trig identity gives:

$$|\text{Cosh}(k\,x+i\,k\,R)| = \text{Cosh}^2(kx)\,\text{Cos}^2(k\,R) + \text{Sinh}^2(kx)\,\text{Sin}^2(k\,R) \ge \text{Cos}^2(k\,R) \qquad (108)$$

Also we have

$$\frac{1}{|x+i\,R|} = \frac{1}{\sqrt{x^2+R^2}} \le \frac{1}{R} \qquad (109)$$

So we have:

$$\frac{1}{2}\int_0^c \frac{e^{x\,t}}{|x+i\,R|\,|\,\text{Cosh}(k\,x+i\,k\,R)|}\,dx \le \frac{c\,e^{c\,t}}{2\,R\,\text{Cos}^2(k\,R)} \qquad (110)$$

This vanishes as $R\to\infty$. A similar proceedure shows that the other horizontal part of the contour integral vanishes. We now show that the integral on the the semicircular part of the contour vanishes (1 point).

$$\left|\int_{C_R} \frac{e^{s\,t}}{2\,s\,\text{Cosh}(k\,s)}\,ds\right| = \left|\int_{\pi/2}^{3\,\pi/2} \frac{e^{R\,e^{i\,\theta}\,t}\,i\,R\,e^{i\,\theta}}{2\,R\,e^{i\,\theta}\,\text{Cosh}(k\,R\,e^{i\,\theta})}\,d\theta\right| \le \frac{1}{2}\int_{\pi/2}^{3\,\pi/2} \frac{e^{R\,t\,\text{Cos}\theta}}{|\text{Cosh}(k\,R\,e^{i\,\theta})|}\,d\theta \qquad (111)$$

A trig identity gives:

$$|\text{Cosh}(k\,R\,e^{i\,\theta})| = \text{Cos}^2(k\,R\,\text{Sin}\,\theta) + \text{Sinh}^2(kR\,\text{Cos}\,\theta) \ge \text{Sinh}^2(kR\,\text{Cos}\,\theta) \qquad (112)$$

For $\theta\neq\pi/2, 3\pi/2$ this grows exponentially as $R\to\infty$. For $\theta=\pi/2$ or $\theta=3\pi/2$ we have

$$|\text{Cosh}(k\,R\,e^{i\,\theta})| = \text{Cos}^2(k\,R) \qquad (113)$$

Which is bounded and non-zero as $R\to 0$ (except for certain special values of R). So we have:

$$\int_{\pi/2}^{3\,\pi/2} \frac{e^{R\,t\,\text{Cos}\theta}}{|\text{Cosh}(k\,R\,e^{i\,\theta})|}\,d\theta \to 0 \qquad (114)$$

by Jordan's Lemma since $\text{Cos}\theta$ is non positive and we just showed that the denominator $\to\infty$ for all θ except the endpoints (which are unimportant to the integral since they are a set of measure zero).
We have shown:

$$\frac{1}{2\,\pi\,i}\int_{c-iR}^{c+iR} \frac{e^{s\,t}}{2\,s\,\text{Cosh}(k\,s)}\,ds = \frac{1}{2\,\pi\,i}\int_{\Gamma_R} \frac{e^{s\,t}}{2\,s\,\text{Cosh}(k\,s)}\,ds \qquad (115)$$

By the residue theorem this last integral is the sum of the residues (1 point)

$$\text{Res}\left(\frac{e^{s\,t}}{2\,s\,\text{Cosh}(k\,s)}, 0\right) + \sum_{n=-\infty}^{\infty} \text{Res}\left(\frac{e^{s\,t}}{2\,s\,\text{Cosh}(k\,s)}, \frac{(2\,n+1)\,\pi}{2\,k}\,i\right) \qquad (116)$$

We now calculate these residues

$$\text{Res}\left(\frac{e^{s t}}{2 s \text{Cosh}(k s)}, 0\right) = \lim_{s\to 0} \frac{s\, e^{s t}}{2 s \text{Cosh}(k s)} = \frac{1}{2}$$

$$\text{Res}\left(\frac{e^{st}}{2 s \text{Cosh}(k s)}, \frac{(2 n+1)\pi}{2 k} i\right) =$$

$$\lim_{s\to(2 n+1)\pi i/2 k} \frac{(s-(2 n+1)\pi i/2 k)\, e^{s t}}{2 s \text{Cosh}(k s)} = \frac{k\, e^{\frac{(2 n+1)\pi i}{2 k} t}}{(2 n+1)\pi i} \lim_{s\to(2 n+1)\pi i/2 k} \frac{s-(2 n+1)\pi i/2 k}{\text{Cosh}(k s)} =$$

$$\frac{e^{\frac{(2 n+1)\pi i}{2 k} t}}{(2 n+1)\pi i} \frac{1}{\text{Sinh}\left(\frac{(2 n+1)\pi i}{2}\right)} = \frac{(-1)^{n+1}}{(2 n+1)\pi} e^{\frac{(2 n+1)\pi i}{2 k} t} \tag{117}$$

So the inverse transform is (1 point)

$$g(t) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(2 n+1)\pi} e^{\frac{(2 n+1)\pi i}{2 k} t} \tag{118}$$

Which by rearranging the terms in the sum and using a trig identity can be written (1 point)

$$g(t) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{(2 n+1)\pi} \text{Cos}\left(\frac{(2 n+1)\pi}{2 k} t\right) \tag{119}$$

(d) (6 points)

We plot the first two terms (2 points)

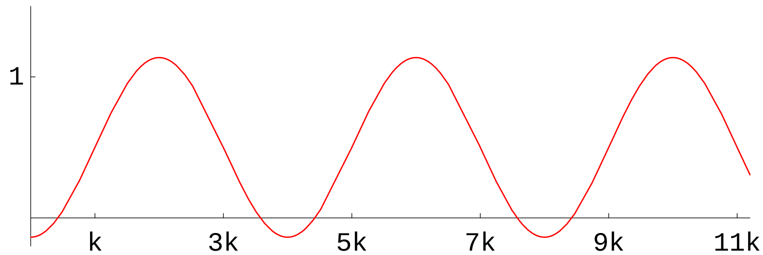

Figure 4

This looks very similar to Figure 2 above.
Notice that the function is periodic (1 point)

$$g(t+4 k) =$$

$$\frac{1}{2} + \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{(2 n+1)\pi} \text{Cos}\left(\frac{(2 n+1)\pi}{2 k}(t+4 k)\right) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{(2 n+1)\pi} \text{Cos}\left(\frac{(2 n+1)\pi}{2 k} t\right) = g(t) \tag{120}$$

which is exactly what we found in part (b).

Since this function has the same periodicity as and a similar appearance to the function found in part (b), we suspect that this series converges to the result in part (b)(1 point).

Other things you might observe. (worth up to 2 points)

Things you may have earned points for:
Showing that the sum converges to a piecewise constant function.
Showing where tthat the minima and maxima of the sum converge to the plateaus of the step function or discussing other geometric aspects of the function the series converges to.
Plotting more terms in the series and discussing the convergence.
Fourier transforming the answer from part (b) to get the answer in part (c).