ACM 95b/100b Handout 3/1/2004: Fourier transforms

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<u>Relation of Fourier transform to Fourier series</u>

Start: Complex form of 2L Fourier series:

(1) $$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n\pi x/L} \qquad \text{on } -L < x < L \quad g \text{ periodic } 2L$$

(2) $$c_n = \frac{1}{2L}\int_{-L}^{L} g(x) e^{-i n\pi x/L}\, dx \qquad \text{For real } g(x) \quad c_k^* = c_{-k}$$

[related to $g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ by $2\,\mathrm{Re}\, c_k = a_k$, $2\,\mathrm{Im}\, c_k = -b_k$

Now let $L \to \infty$ so we can represent nonperiodic functions on infinite interval

As $L \to \infty$ look at (2): if the integral is to exist $g(x) \to 0$ as $x \to \pm\infty$

So most of the contribution to the integral comes from some finite range of x - say $-M < x < M$. If we choose M large enough we can get any accuracy we want.

So now consider some finite n in (2) as $L \to \infty$

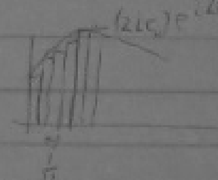
$$2L\, c_n \to \int_{-M}^{M} g(x)\cdot 1\, dx \qquad \text{since } e^{-i n x \pi/L} \to 1 \text{ as } L \to \infty \text{ for any finite } x, n.$$

So this is not interesting.

Instead let $f = \frac{n}{2L}$ and keep f finite as $L \to \infty$ (i.e. let $n \to \infty$)

Then as $L \to \infty$ $$(2L c_n) \to \int_{-\infty}^{\infty} g(x) e^{-i2\pi f x}\, dx$$

and (1) $\Rightarrow$ $$g(x) = \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} (2L c_n) e^{i2\pi f x} \underbrace{\frac{1}{2L}}_{\Delta n = 1}$$



$$= \int_{-\infty}^{\infty} (2L c_n) e^{i2\pi f x}\, df \qquad \text{since } \Delta f = \frac{\Delta n}{2L}$$

Thus if we identify

$$\tilde{g}(f) = 2L c_n \qquad f = \frac{n}{2L}$$

we have

$$\tilde{g}(f) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi f x} dx \qquad \text{Fourier transform}$$

and $$g(x) = \int_{-\infty}^{\infty} \tilde{g}(f) e^{i2\pi f x} df \qquad \text{Inverse transform}$$

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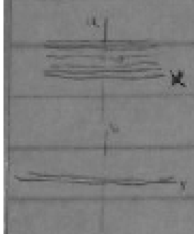
Proof of Convolution Theorem for Fourier transforms

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du \qquad \text{convolution}$$

$$FT(f * g) = \int_{-\infty}^{\infty} e^{-i2\pi f x}\, dx \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, e^{-i2\pi f x}\, dx\, du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u) e^{-i2\pi f u}\, g(x-u)\, e^{-i2\pi f (x-u)}\, dx\, du$$



Let $s = x - u$ — still runs $-\infty \to \infty$ $\quad ds = dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u) e^{-i2\pi f u}\, g(s)\, e^{-i2\pi f s}\, ds\, du$$

$$= \int_{-\infty}^{\infty} f(u) e^{-i2\pi f u}\, du \int_{-\infty}^{\infty} g(s)\, e^{-i2\pi f s}\, ds$$

$$FT(f * g) = \underset{\underset{FT(f)}{\uparrow}}{\tilde{f}} \cdot \underset{\underset{FT(g)}{\uparrow}}{\tilde{g}}$$