ACM 95b/100b Final Exam

March 12, 2004 | Due March 17, 2004
ACM 95b/100b | 3pm in Firestone 307
E. Sterl Phinney | (2 pts) Include grading section number

The honor code is in effect. Please follow all of the following instructions regarding this exam. If you feel unclear about any of these instructions, you are required by the honor code to ask for clarification.

- You may not communicate with any other person (except Prof. Phinney) about the contents of this exam, until both you and the other person (if taking ACM95b/100b) have submitted your exams. The Deans have given a few students extensions on the exam, so please do not talk about it loudly in public, even after the due date.
- The exam must be completed in a single sitting of **4 hours**.
- There are 5 problems, with point values (24, 10, 24, 24, 16) respectively. The point values for each subpart are indicated at the start of each problem. The total exam is worth $24+10+24+24+16+2 = 100$ points (the 2 points are for writing your grading section number on the cover of your blue book).
- Calculators, computers and related devices are **not permitted**.
- **Closed book, computer off exam:** with the exception of official lecture handouts, problem set questions, *and official problem set solutions* from this course (2004 ACM95b/100b), only material written in your own hand may be used during the exam.
- Please write your exam in standard **blue books** and make sure your name and **grading section number** (as assigned by the ACM95b/100b Underground) is clearly written on the front of each blue book.
- You must sign in your completed exam in person to Maria Katsas in Firestone 307 by **3pm on Wednesday March 17**.

Good luck!

"That's some of my earlier work."

The artist (and the cartoonist, Sidney Harris) must have taken ACM95b/100b, since he knows about $T_n(x)$ [see problem 1d]!

Please be sure to write your (Underground-assigned) grading section number on the cover of your bluebook, for 2 free points.

The exam starts on the next page.

1. (4×6 points)

a) Classify each of the following as either a Sturm-Liouville problem, or not. If it is a Sturm-Liouville problem, further classify it as regular, periodic or singular.

$$y'' + \lambda x y = 0, \qquad y(0) = 0, \quad y'(3) + y(3) = 0 \tag{1}$$

$$x^2 y'' + 2xy' + (x^2 - \lambda)y = 0, \qquad y(0) = 1, \quad y'(6) + 2y(6) = 0 \tag{2}$$

$$(1 - x^7)y'' - 7x^6 y' + \lambda x^7 y = 0, \qquad y(0) = 0, \quad y(1) = \text{finite} \tag{3}$$

$$y'' + [\lambda - 20\cos(2x)]\, y = 0, \qquad y(x) = y(x + 2\pi), \quad y'(x) = y'(x + 2\pi) \tag{4}$$

$$\frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) + \lambda y = 0, \qquad y(1) = 0, \quad y(e) = 0 \tag{5}$$

$$y'' + 20(\operatorname{sech} x)^2 y + \lambda y = 0, \qquad y \to 0 \text{ as } |x| \to \infty \tag{6}$$

b) Which one of the equations of part (a) arises as the radial equation in separation of variables of $\nabla^2 u(r, \theta, \phi) = 0$ in spherical coordinates (don't consider boundary conditions):

$$0 = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \ ? \tag{7}$$

c) By finding a multiplying factor, put the following into Sturm-Liouville form:

$$(1 - x^2)y'' - xy' + \lambda y = 0, \quad y(-1) = \text{finite}, \quad y(1) = \text{finite} \tag{8}$$

d) Referring to your Sturm-Liouville form of eq (8):

i. What is the orthogonality condition obeyed by the eigenfunctions (call them $T_n(x)$) of eq (8) with distinct eigenvalues?

ii. Could the eigenfunctions form a complete basis (for continuous, square-integrable functions)? Why or why not?

iii. Are they guaranteed to be complete by theorems given in class?

2. (6 + 4 points)

a) Determine the eigenvalues λ_n and corresponding normalized eigenfunctions $y_n(x)$ for

$$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \lambda y = 0, \qquad y(1) = 0, \quad y(e) = 0 \tag{9}$$

[hints: try solutions of the form $y = x^\nu$. Things may simplify if you define $4\lambda - 1 = \mu^2$. Remember that $x^{ib} = \exp(ib\ln x)$. As a check, you should find that $\lambda_4 = 16\pi^2 + 1/4$.]

b) By explicit calculation, verify that the following theorems about the eigenvalues and eigenfunctions of a regular Sturm-Liouville problem apply to your solutions of part (a):

i. There is an infinite number of discrete eigenvalues with no accumulation point ($\lambda_n \to \infty$ as $n \to \infty$).

ii. There is a unique (up to a multiplicative factor) eigenfunction corresponding to each eigenvalue.

iii. The eigenfunctions are orthogonal.

iv. The eigenfunctions are complete. [hint: do this by showing a relationship to another familiar set of basis functions known to be complete].

3. $(4 \times 6$ points)

a) Find the Green's function $G(x;t)$ for the differential operator

$$Ly = \frac{d}{dx}\left(x\frac{dy}{dx}\right), \quad y(0) = \text{finite}, \quad y(1) = 0 \tag{10}$$

(i.e. $G(x;t)$ is the solution to $LG = \delta(x-t)$ for $0 < t < 1$, $0 \le x \le 1$) and give the solution to the BVP

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = f(x), \quad y(0) = \text{finite}, \quad y(1) = 0 \tag{11}$$

as an integral involving $G(x,t)$ and $f(t)$.

b) Verify that your Green's function correctly gives the solution for $f(x) = 1$, which is easily verified[1] to be $y(x) = x - 1$? [hint: $\int \ln t \, dt = t\ln(t) - t$]

c) Find the Green's function $g(x,t)$ for the differential operator

$$Ly = y'' + y, \quad y'(0) = 0, \quad y(1) = 0 \tag{12}$$

(i.e. $g(x,t)$ is the solution to $Lg = \delta(x-t)$ for for $0 < t < 1$, $0 \le x \le 1$), and give the solution to the BVP

$$y'' + y = f(x), \quad y'(0) = 0, \quad y(1) = 0\ . \tag{13}$$

on $0 \le x \le 1$ as an integral involving $g(x,t)$ and $f(t)$.

d) If we modify the right boundary condition of the previous part to $y(\pi/2) = 0$, does there exist a solution on $0 \le x \le \pi/2$ to the BVP

$$y'' + y = f(x), \quad y'(0) = 0, \quad y(\pi/2) = 0\ . \tag{14}$$

for arbitrary continous functions $f(x)$? Why or why not?

[1] If you get agreement, you can be proud of yourself: in the 3rd edition of Arfken, he didn't get agreement, and devoted a whole page to a 'rigorous' (and incorrect) proof [p. 906-907] of why it shouldn't work. This problem illustrates that Green's functions can work even with boundary conditions at a singular point.

4. (4×6 points)

Your sweetie gives you a thin metal ring of circumference 2π in some units. Unfortunately the jeweler who made it confused plutonium with palladium, and also didn't mix the alloy well, so the plutonium fraction is not uniform around the ring. You notice the ring getting kind of hot, and want to figure out how it heats up.

Let $u(x, t)$ describe the temperature of the ring, where $-\pi < x \leq \pi$ is the circumferential coordinate, and t is time. Because of the plutonium, the ring has nonuniform internal heating, and its temperature is therefore described by

$$u_t = \kappa u_{xx} - \sigma u + f(x) \tag{15}$$

where $\kappa > 0$ is the constant thermal diffusivity, $\sigma > 0$ is the constant coefficient of heat loss to your finger, and $f(x)$ describes the heating by plutonium decay. Suppose $u(x, 0) = 0$.

a) Because it is defined on a ring, the temperature must be a 2π periodic function of x. This suggests that you should write the temperature as Fourier expansion with time-dependent coefficients,

$$u(x, t) = \sum_{n=-\infty}^{\infty} A_n(t) e^{inx} \tag{16}$$

You may also assume $f(x)$ can be represented by a similar Fourier series (with *time-independent* coefficients). Find the ordinary differential equations and initial conditions obeyed by each $A_n(t)$.

b) Solve the equations you found in part (a) for each $A_n(t)$.

c) Suppose $f(x) = \cos^2(x)$. Find an explicit solution for $u(x, t)$. [hint: $2\cos^2 x = (1 + \cos(2x))$, and only three of the A_n are nonzero.]

d) Something goes wrong with the solutions you found in the previous two parts as $\sigma \to 0$. Find the correct solution when $\sigma = 0$ for general $f(x)$ (i.e. part (b)), and for the particular case $f(x) = \cos^2(x)$ (part(c)), and explain the results physically.

5. $(6 + 5 + 5$ points)

The soil has a periodically varying surface temperature $\mathcal{T}(0,t)$ (modulated on a 24-hour cycle, and also on a 1-year seasonal cycle). How does the soil temperature $\mathcal{T}(z,t)$ vary below the surface[2]? To simplify the equations, we let $\mathcal{T}_0$ be the mean surface temperature, let the soil surface be at $z = 0$, measure depth below the surface with coordinate $z > 0$, let the soil's thermal conductivity κ be constant, and define

$$T(z,t) \equiv \mathcal{T}(z,t) - \mathcal{T}_0 \ . \tag{17}$$

Then the desired $T(z,t)$ is governed by the heat equation and the specified boundary conditions [3] for $-\infty < t < \infty$:

$$\begin{aligned} \frac{\partial T}{\partial t} &= \kappa \frac{\partial^2 T}{\partial z^2} \ , & (18) \\ T(0,t) &= \Delta T \sin \omega_0 t \ , & (19) \\ T(\infty,t) &= 0 \ . & (20) \end{aligned}$$

Take $\omega_0 > 0$ and ΔT constant.

a) Fourier transform in time equation eq (18) and its boundary conditions, to derive the equation for the Fourier transform $\hat{T}(z,f)$ of $T(z,t)$, and the Fourier transforms of the boundary conditions $\hat{T}(0,f)$ and $\hat{T}(\infty,f)$.

b) Solve your equations of part (a) for $\hat{T}(z,f)$ (hint: make sure you treat positive and negative Fourier frequencies separately, and check that your solution satisfies the boundary condition in each case).

c) Do the inverse Fourier transform to find $T(z,t)$. You should find[4] that

$$T(z,t) = \mathcal{T} - \mathcal{T}_0 = \Delta T \exp\left(-z\sqrt{\omega_0/(2\kappa)}\right) \sin\left(\omega_0 t - z\sqrt{\omega_0/(2\kappa)}\right) \ . \tag{21}$$

[2]The answer, which you will derive here, is of considerable importance to desert fauna, hibernating animals, the Alaskan and Siberian construction industry, and the designers of the thermal storage system under construction outside Braun gym.

[3]$\omega_0 = 2\pi/(24 \text{ hours}) \equiv \omega_1$ is of primary interest to desert animals, and $\omega_0 = 2\pi/(1 \text{ year}) \equiv \omega_2$ is of primary interest to housing contractors in permafrost regions.

[4]After the exam, you may be interested to put $\kappa \sim 3 \times 10^{-7} \text{m}^2\text{s}^{-1}$ for rock and packed soil, and find the depths in meters at which the temperature variation is reduced to $1/e$ of its surface value for the daily and for the annual temperature cycles. You can also check that when the surface BC of equation (19) is replaced by the real-world simultaneious sum of both cycles, $T(0,t) = \Delta T_1 \sin \omega_1 t + \Delta T_2 \sin(\omega_2 t + \phi_2)$, that you can get the solution for this case by simple linear superposition of two solutions like eq (21). You might also like to think about the origin of the sinusoid: why are there depths that are colder in daytime/summer than at night/winter (these layers can actually be measured at stably stratified ocean depths)? Could you design a system that makes some interesting use of these?