‘Magnetic’ Components of Gravitational Waves and Their Effect in the LIGO-VIRGO Response Functions

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(For more details see D. Baskaran and L. P. Grishchuk, CQG **21**, 4041 (2004))

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Content

- Motion of free charges in the field of an electromagnetic wave. Electric and magnetic components of the Lorentz force
- Weak plane gravitational waves in the local inertial (Lorentzian) frame
- Motion of free masses in the field of a gravitational wave . `Electric' and `magnetic' components of motion. Zero-order and first-order terms in **L/λ**
- Equations of motion of test masses from the geodesic deviation equation
- `Magnetic' contribution to the variation of the distance between test masses
- Response of a 2-arm interferometer to the incoming plane gravitational wave
- `Magnetic' modifications in the angular and frequency dependencies of the LIGO-VIRGO response functions
- Astrophysical example. The size of the `magnetic' contributions
- The urgent need for taking `magnetic' components into account in the existing data analysis pipelines

Motion of a charged particle in the field of an electromagnetic wave

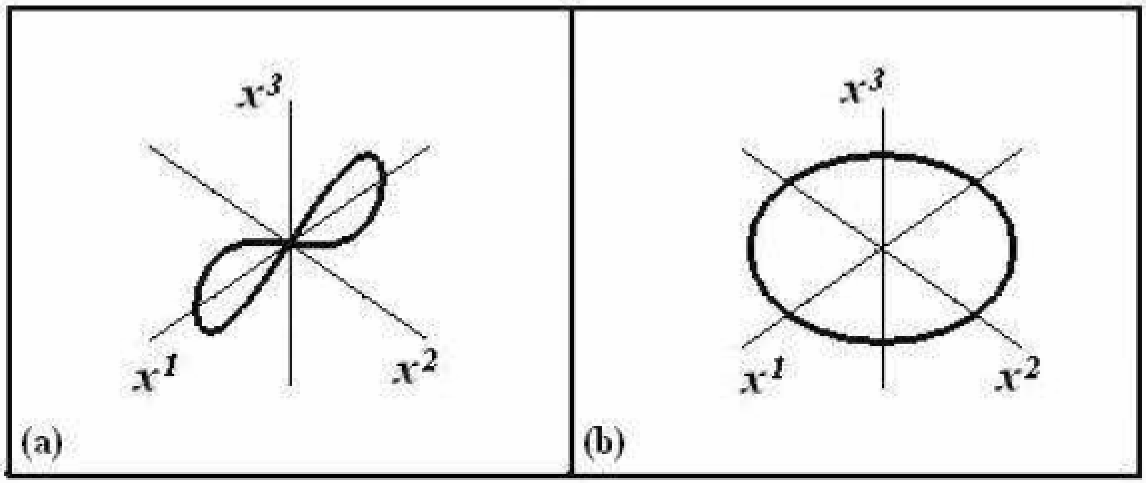

Figure 1. The figure a) on the left shows the trajectory of a charged particle in the field of a linearly polarized electromagnetic wave, whereas the figure b) on the right shows the trajectory in the field of a circularly polarized wave.

Electromagnetic (Lorentz) force: $m\frac{d^2\mathbf{x}}{dt^2} = e\mathbf{E} + \frac{e}{c}\left(\frac{d\mathbf{x}}{dt} \times \mathbf{H}\right).$

In the field of a weak electromagnetic wave, size of the x^3 -amplitude of oscillations is small in comparison with the x^1 -amplitude:

$$x^3 \sim x^1 \left(\frac{x^1}{\lambda}\right)$$

A general weak plane gravitational wave:

$$ds^2 = c^2 dt^2 - [\delta_{ij} + h_{ij}]\, dx^i dx^j.$$

$$h_{ij} = \overset{1}{p}_{ij} a + \overset{2}{p}_{ij} b, \qquad \overset{1}{p}_{ij} = m_i m_j - n_j n_i, \quad \overset{2}{p}_{ij} = -m_i n_j - m_j n_i,$$

where

$$a = h_+ \sin\left(k(x^0 + x^3) + \psi_+\right), \quad b = h_\times \sin\left(k(x^0 + x^3) + \psi_\times\right),$$

In the frame based on principal axes (using rotation in the (x^1, x^2) plane):

$$ds^2 = c^2 dt^2 - (1+a) {dx^1}^2 - (1-a) {dx^2}^2 + 2b\, dx^1 dx^2 - {dx^3}^2,$$

$$a = h_+ \sin\left(k(x^0 + x^3) + \psi\right), \quad b = h_\times \cos\left(k(x^0 + x^3) + \psi\right),$$

Two linear polarizations: $h_+ = 0$ or $h_\times = 0$

two circular polarizations: $h_+ = h_\times$ or $h_+ = -h_\times$

Coordinate transformation to a local inertial coordinate system; the closest thing to a global Lorentzian coordinate system used in the electromagnetic example (Grishchuk1977):

$$\bar{x}^0 = x^0 + \tfrac{1}{4}\dot{a}\,\left({x^1}^2 - {x^2}^2\right) - \tfrac{1}{2}\dot{b}\,x^1 x^2,$$

$$\bar{x}^1 = x^1 + \tfrac{1}{2}a\,x^1 - \tfrac{1}{2}b\,x^2 + \tfrac{1}{2}\dot{a}\,x^3 x^1 - \tfrac{1}{2}\dot{b}\,x^3 x^2,$$

$$\bar{x}^2 = x^2 - \tfrac{1}{2}a\,x^2 - \tfrac{1}{2}b\,x^1 - \tfrac{1}{2}\dot{a}\,x^3 x^2 - \tfrac{1}{2}\dot{b}\,x^3 x^1,$$

$$\bar{x}^3 = x^3 - \tfrac{1}{4}\dot{a}\,\left({x^1}^2 - {x^2}^2\right) + \tfrac{1}{2}\dot{b}\,x^1 x^2.$$

(You cannot proceed without quadratic terms, but there is no need for higher-order terms)

Trajectories of the nearby free particles $x^i = l^i$ including their `magnetic' oscillations back and forth in the direction of the wave propagation, i.e. the x^3 -direction:

$$\bar{x}^1(t) = l_1 + \tfrac{1}{2}\left[h_+ l_1 \sin(\omega t + \psi) - h_\times l_2 \cos(\omega t + \psi)\right]$$

$$+ \tfrac{1}{2}k\left[h_+ l_3 l_1 \cos(\omega t + \psi) + h_\times l_3 l_2 \sin(\omega t + \psi)\right],$$

$$\bar{x}^2(t) = l_2 - \tfrac{1}{2}\left[h_+ l_2 \sin(\omega t + \psi) + h_\times l_1 \cos(\omega t + \psi)\right]$$

$$- \tfrac{1}{2}k\left[h_+ l_3 l_2 \cos(\omega t + \psi) - h_\times l_3 l_1 \sin(\omega t + \psi)\right],$$

$$\bar{x}^3(t) = l_3 - \tfrac{1}{4}k\left[h_+\left({l_1}^2 - {l_2}^2\right)\cos(\omega t + \psi) + 2h_\times l_1 l_2 \sin(\omega t + \psi)\right].$$

Familiar picture of the deformations of a disk consisting of free particles. Only zero-order approximation in the ratio L/λ shown, `magnetic' contribution ignored:

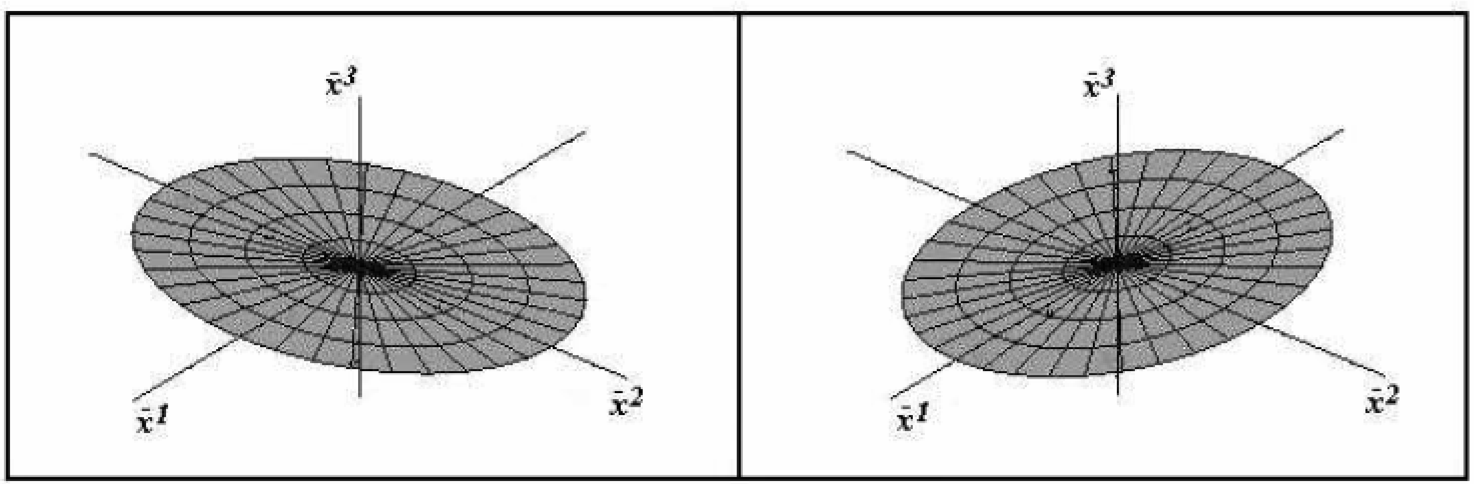

Figure 3. Deformation of a disk of free test particles in the field of a linearly polarized ($h_+ \neq 0, h_\times = 0$) g.w. in the limit of $k = 0$. The two figures show the displacements at the moments of time separated by a half period.

Deformations of the disk with the terms proportional to the wavenumber k, i.e. L/λ `magnetic' contributions, included:

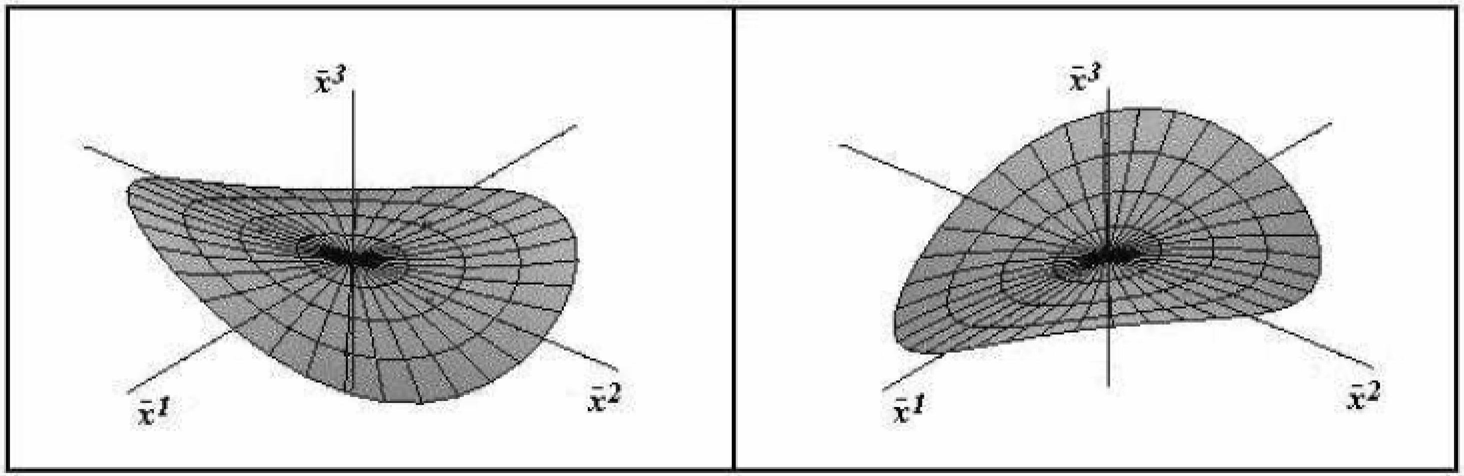

Figure 5. The figure shows the deformations of a circular disk of free test particles under the action of a linearly polarized g.w. $(h_+ \neq 0, h_\times = 0)$. The "magnetic" contribution is responsible for the displacements along the $\bar{x}^3$ axis. The two pictures show the configurations at the moments of time separated by a half of period.

Equations of motion from the geodesic deviation equation

A two-parameter family of timelike geodesics: $x^\mu(\tau, r)$

A tangent vector to the geodesic line, and the separation vector between geodesics:

$$u^\mu(\tau, r) = \frac{\partial x^\mu}{\partial \tau}\Big|_{r=const} \; , \qquad n^\mu(\tau, r) = \frac{\partial x^\mu}{\partial r}\Big|_{\tau=const} \; .$$

The central geodesic is at $r = 0$ a nearby geodesic is at $r = r_0$

For small r_0 :

$$x^\mu(\tau, r_0) = x^\mu(\tau, 0) + r_0 \frac{\partial x^\mu}{\partial r} + \frac{1}{2} r_0^2 \frac{\partial^2 x^\mu}{\partial r^2} + O(r_0^3).$$

In the lowest approximation in terms of r_0 the geodesic deviation equation has the form (Misner, Thorne, Wheeler 1973):

$$\frac{D^2 n^\mu}{d\tau^2} = R^\mu_{\alpha\beta\gamma} u^\alpha u^\beta n^\gamma,$$

In the next approximation we derive (based on Bazanski 1977):

$$\frac{D^2 N^\mu}{d\tau^2} = R^\mu_{\alpha\beta\gamma} u^\alpha u^\beta N^\gamma + \frac{1}{2}\left(R^\mu_{\alpha\beta\gamma;\delta} - R^\mu_{\gamma\delta\alpha;\beta}\right) u^\alpha u^\beta N^\gamma N^\delta + 2R^\mu_{\alpha\beta\gamma} u^\beta \frac{DN^\alpha}{d\tau} N^\gamma + O(r_0^3).$$

where

$$x^\mu(\tau, r_0) = x^\mu(\tau, 0) + N^\mu - \Gamma^\mu_{\alpha\beta} N^\alpha N^\beta + O(r_0^3).$$

In the local inertial frame :

$$\frac{d^2 N^i}{dt^2} = R^i_{00j} N^j + \frac{1}{2}\left(R^i_{00j;k} - R^i_{jk0;0}\right) N^j N^k + 2R^i_{j0k} \frac{dN^j}{dt} N^k$$

Specifically in the field of a plane gravitational wave (and $N^i(t) = l^i + \xi^i(t)$):

$$\frac{d^2\xi^i}{dt^2} = \frac{1}{2}\omega^2 l^j \left[\overset{1}{p}{}^i_{\ j} h_+ \sin(\omega t + \psi) + \overset{2}{p}{}^i_{\ j} h_\times \cos(\omega t + \psi)\right]$$
$$- \frac{1}{2}\omega^2 l^k l^l \left[k_l \delta^{ij} + \frac{1}{2} k^i \delta^j_l\right] \left[\overset{1}{p}{}_{kj} h_+ \cos(\omega t + \psi) - \overset{2}{p}{}_{kj} h_\times \sin(\omega t + \psi)\right].$$

The gravitational-wave force consists of the `electric' and `magnetic' parts:

$$m\frac{d^2\xi^i}{dt^2} = F^i_{(e)} + F^i_{(m)}.$$

The `magnetic' force is proportional to the particle's velocity:

$$\frac{F^i_{(m)}}{m} = \omega l^l \left[k_l \delta^i_j + \frac{1}{2} k^i \delta_{jl}\right] \frac{d\xi^j}{dt}.$$

Solutions to the geodesic deviation equations of motion coincide exactly with trajectories derived by the transformation from the synchronous to the local inertial frame

[Note that in the gravitational-wave case the term $\frac{1}{2}\left(R^i_{00j;k} - R^i_{jk0;0}\right) N^j N^k$

is linear in **h** (and was retained) whereas the term $2R^i_{j0k}\frac{dN^j}{dt}N^k$

is quadratic in **h** (and was neglected). The concept of `gravitomagnetism', on the

contrary, uses the second term and neglects the first one (assumes large initial

velocity $\frac{dN^j}{dt}$).]

Variation of the distance between test masses

Variation of the distance between the central particle (corner mirror of the interferometer) and a nearby particle (end mirror of the interferometer) depends on the orientation of the interferometer's arm and frequency of the wave:

$$d(t) = l + \frac{1}{2l}\left[h_+({l_1}^2 - {l_2}^2)\sin(\omega t + \psi) - 2h_\times l_1 l_2 \cos(\omega t + \psi)\right]$$

$$+ \frac{1}{4l} k l_3 \left[h_+({l_1}^2 - {l_2}^2)\cos(\omega t + \psi) + 2h_\times l_1 l_2 \sin(\omega t + \psi)\right] + O\left(hl(l/\lambda)^2\right).$$

Exact formula based on the time difference between photon's departure and return:

$$d(t) = l + \frac{1}{2l}\left[h_+ \frac{l_1^2 - l_2^2}{2}\Phi_c + h_\times l_1 l_2 \Phi_s\right]$$

where

$$\Phi_c = -\frac{1}{k(l + l_3)}\left[\cos(\omega t + \psi + k l_3) - \cos(\omega t + \psi - k l)\right]$$
$$+ \frac{1}{k(l - l_3)}\left[\cos(\omega t + \psi + k l_3) - \cos(\omega t + \psi + k l)\right]$$

gives precisely the above result when expanded up to L/λ terms.

Response of a 2-arm interferometer: $\Delta d(t) = d(t)^{(1)} - d(t)^{(2)}$

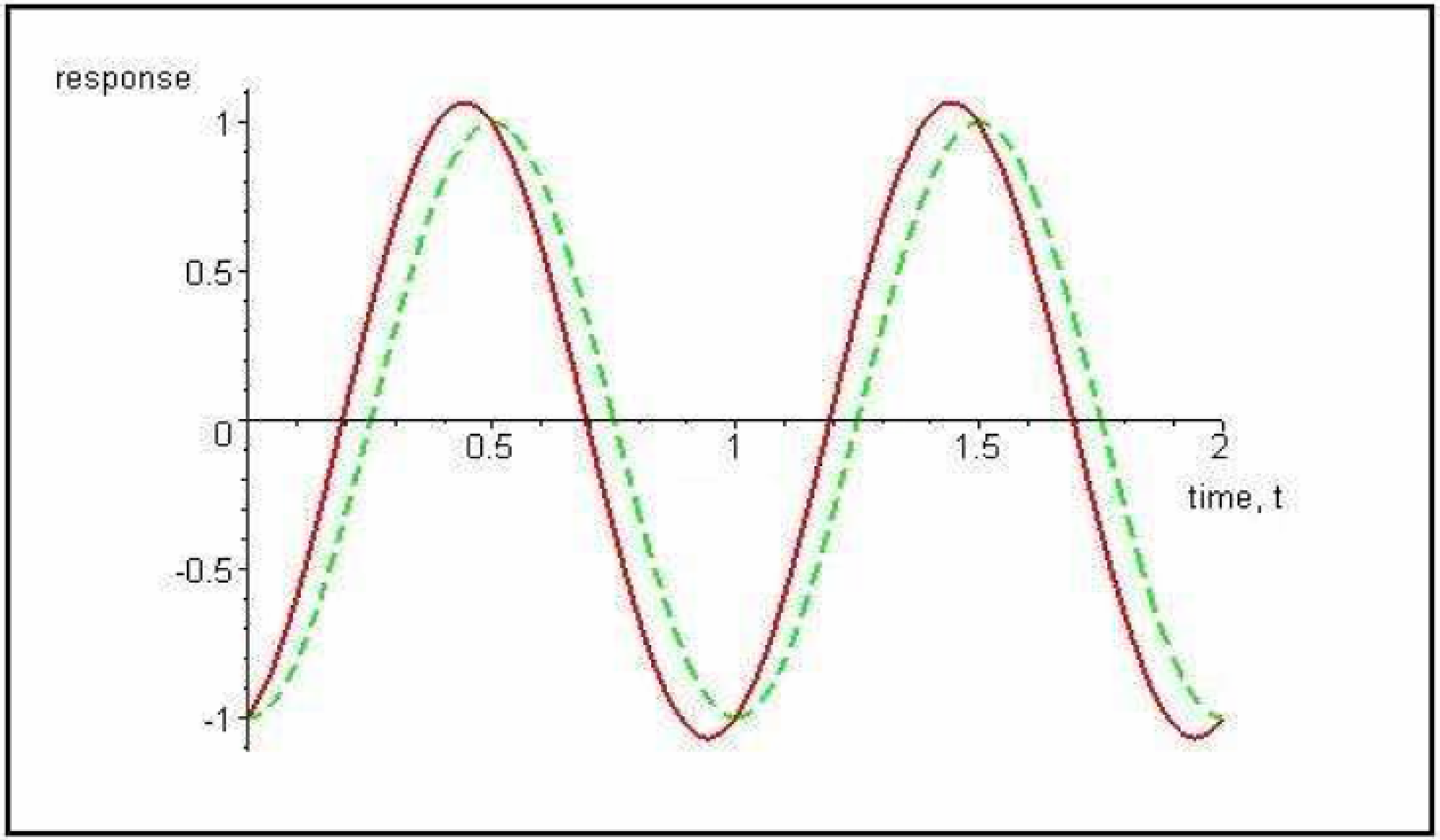

Figure 8. A typical response of an interferometer, as a function of time, to the monochromatic circularly polarized gravitational wave coming from a fixed direction on the sky. The solid line shows the total response, while the dashed line is purely "electric" part.

Angular pattern for a fixed polarization and a given frequency of the wave

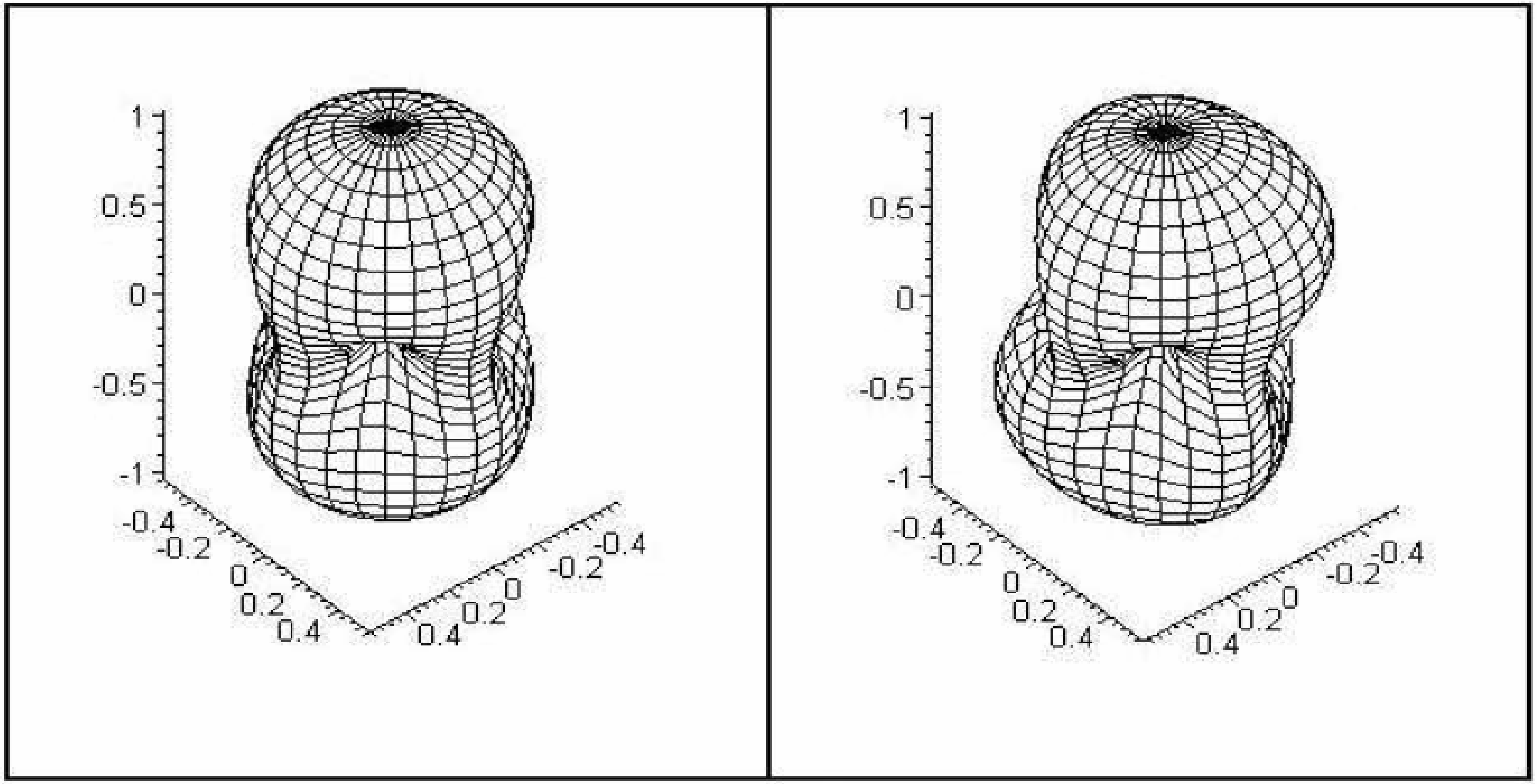

Figure 9. The amplitude of the interferometer's response to circularly polarized waves. The graphs are normalized in such a way that the amplitude is equal 1 for $\Theta = 0$. The left figure ignores the "magnetic" effect, whereas the right figure shows the total response.

Astrophysical example: a pair of stars on a circular orbit in a plane orthogonal to the line of sight.

$$h_+ = h_\times = h_R = \frac{32\pi^2 G}{Rc^4} M r^2 \omega^2,$$

Correct response of the interferometer, including its `magnetic' part:

$$\Delta d(t) = l h_R \left[\left\{ \cos 2\Phi \left(\frac{1+\cos^2\Theta}{2} \right) - \frac{1}{4}\frac{\omega l}{c} \sin 2\Phi \left(\cos\Phi + \sin\Phi\right) \cos\Theta \sin\Theta \right\} \right.$$
$$\times \sin(\omega t + \psi)$$
$$- \left\{ \sin 2\Phi \cos\Theta - \frac{1}{4}\frac{\omega l}{c} \left(\cos^2\Theta + \sin 2\Phi \left(\frac{1+\cos^2\Theta}{2} \right) \right) (\cos\Phi - \sin\Phi) \sin\Theta \right\}$$
$$\left. \times \cos(\omega t + \psi) \right].$$

Response based on the `electric' contribution only (incorrect):

$$\Delta d(t) = l\ h_R \left[\cos 2\Phi \left(\frac{1+\cos^2\Theta}{2} \right) \sin(\omega t + \psi) - \sin 2\Phi \cos\Theta \cos(\omega t + \psi) \right].$$

Conclusions

The output data D are related to the astrophysical signal S through the response function R of the instrument: D=RS. Sophisticated theoretical templates will be wasted if the response function is not known with the equally high accuracy. There will be no accurate astrophysics without accurate response function !

In the LIGO-VIRGO interferometers, the `magnetic' component of the g.w. force, proportional to (kl), provides corrections to the interferometer's response at the level of 5 percent in the frequency band of 600 Hz, and up to 10 percent in the frequency band of 1200 Hz. Corrections are not a `mis-calibration number', they are complicated functions of observational direction, as well as polarization and frequency of the incoming waves !

Data analysis based on the `electric' contribution only can significantly compromise the determination of parameters of the g.w. source.

`Magnetic' prediction of general relativity is important, measurable, and it must be taken into account in accurate analysis of a variety of astrophysical sources of gravitational waves.