R-Mode Oscillations in Spinning Neutron Stars: Are Their Gravitational Waves Detectable by LIGO?

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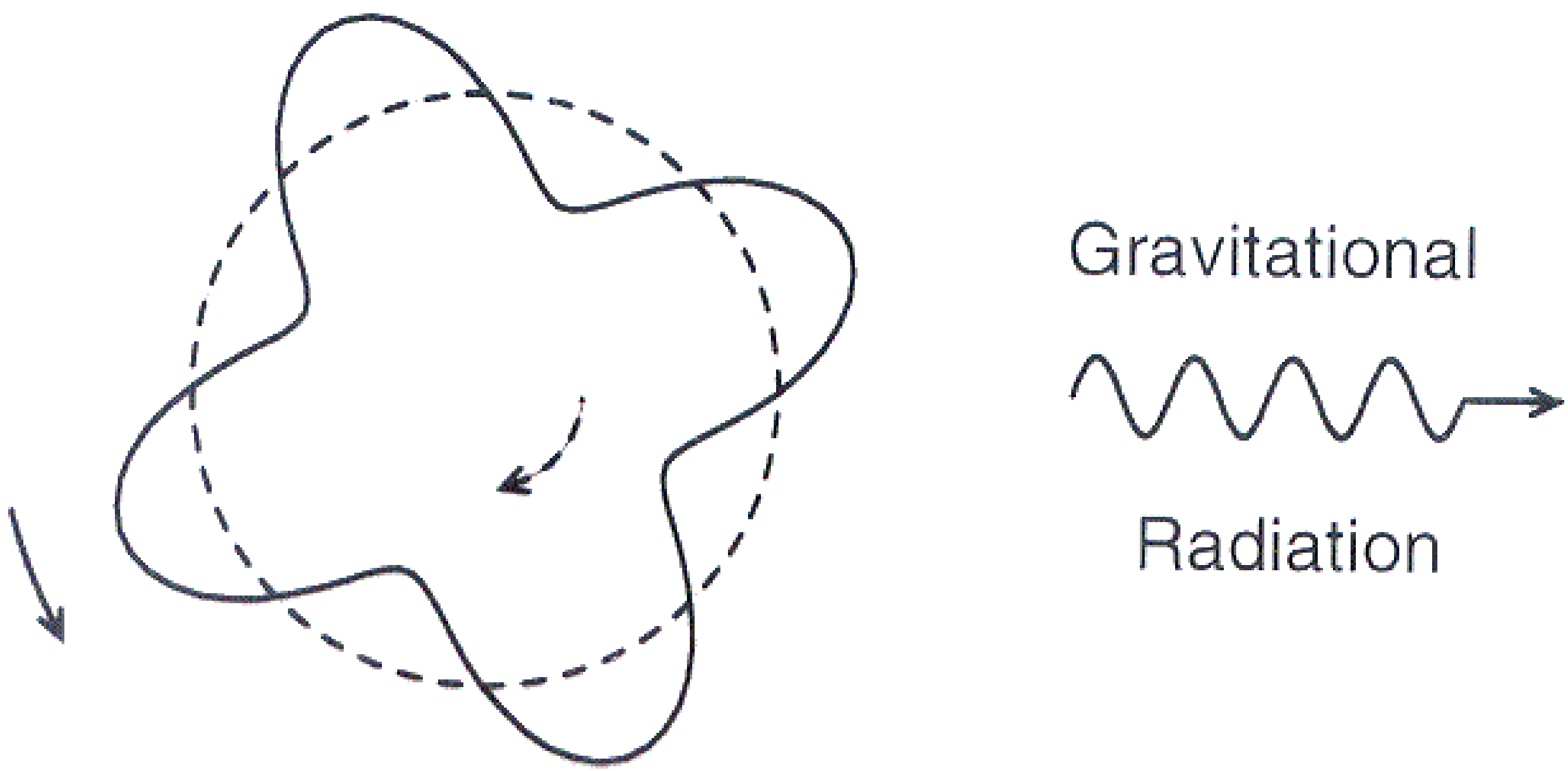
- The non-radial pulsations of neutron stars couple to gravitational radiation.
- Back reaction to the emission of gravitational radiation drives an instability in the counter-rotating modes of spinning neutron stars.
- Gravitational radiation removes angular momentum and thus reduces the angular velocities of young neutron stars.
- Gravitational radiation emitted from these unstable modes may be detectable by LIGO.

Pulsations in Rotating Stars may be driven unstable by Gravitational Radiation



- Waves moving in the opposite direction as the star (as seen by an observer at rest on the star) have negative angular momentum.

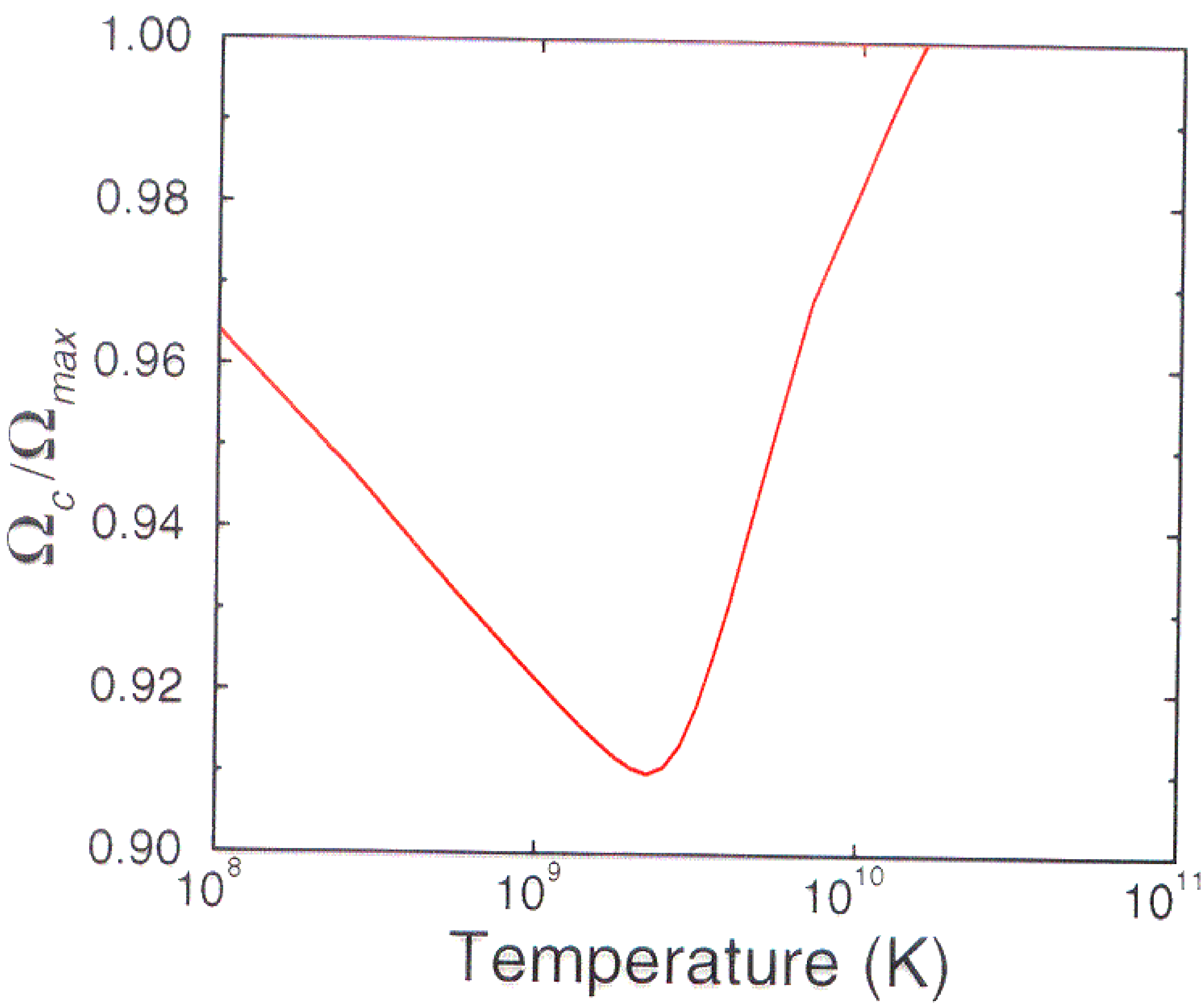
- Waves moving in the same direction as the star (as seen from infinity) emit positive angular momentum.

- To conserve angular momentum such waves must grow: they emit positive angular momentum into gravitational radiation, hence their own negative angular momentum must decrease.

- Viscosity suppresses the gravitational radiation driven instability in the f-modes except in the most rapidly rotating stars.

- Stars with $\Omega < \Omega_c$ are stable.

- Stars with $\Omega_c < \Omega < \Omega_{max}$ are unstable.



- The r-modes of rotating stars are primarily surface currents driven by Coriolis forces.

$$\delta\vec{v} = \alpha R\Omega \left(\frac{r}{R}\right)^m \vec{r} \times r\vec{\nabla} Y_{mm}\, e^{i\omega t} + \mathcal{O}(\Omega^3)$$

- The density perturbations associated with these modes vanish at lowest order in the angular velocity: $\delta\rho = \mathcal{O}(\Omega^2)$.

- The frequencies of the r-modes are

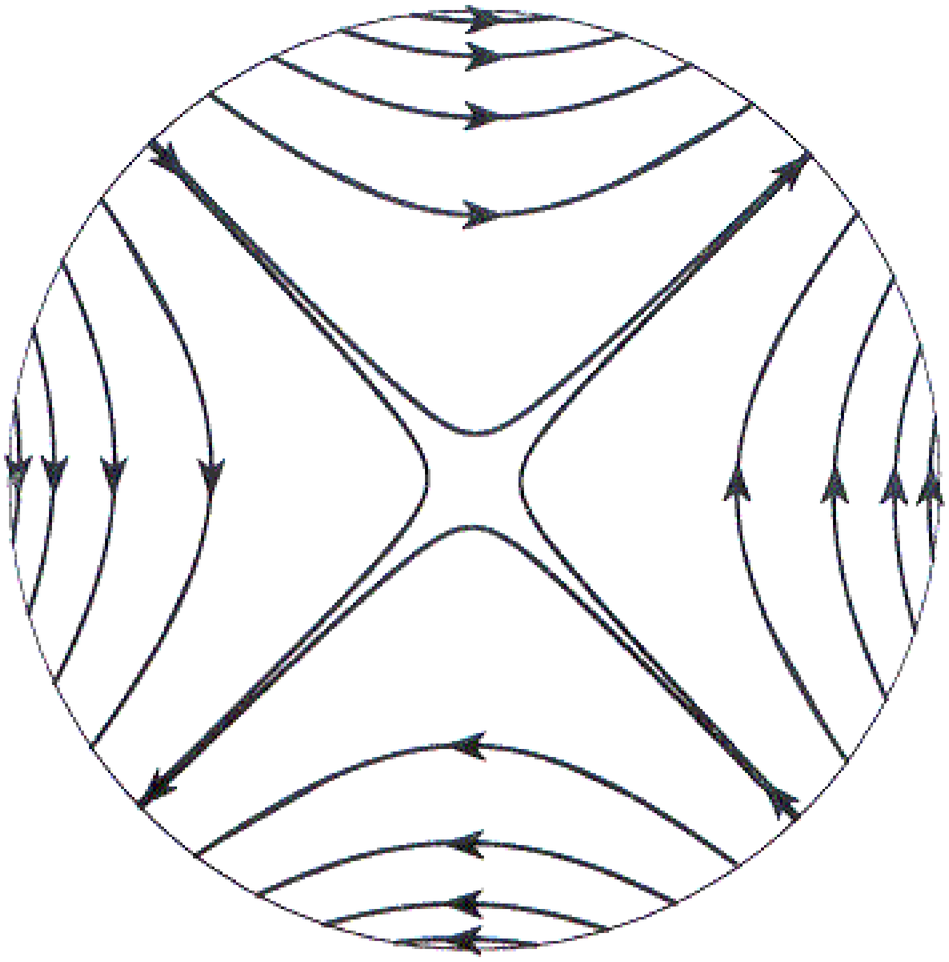
$$\omega = -\frac{(m-1)(m+2)}{m+1}\Omega + \mathcal{O}(\Omega^3).$$

- The flow pattern of the r-modes moves in the inertial frame with angular velocity $-\omega/m$ ($\frac{2}{3}\Omega$ for the $m=2$ case) in the same direction as the star's rotation.

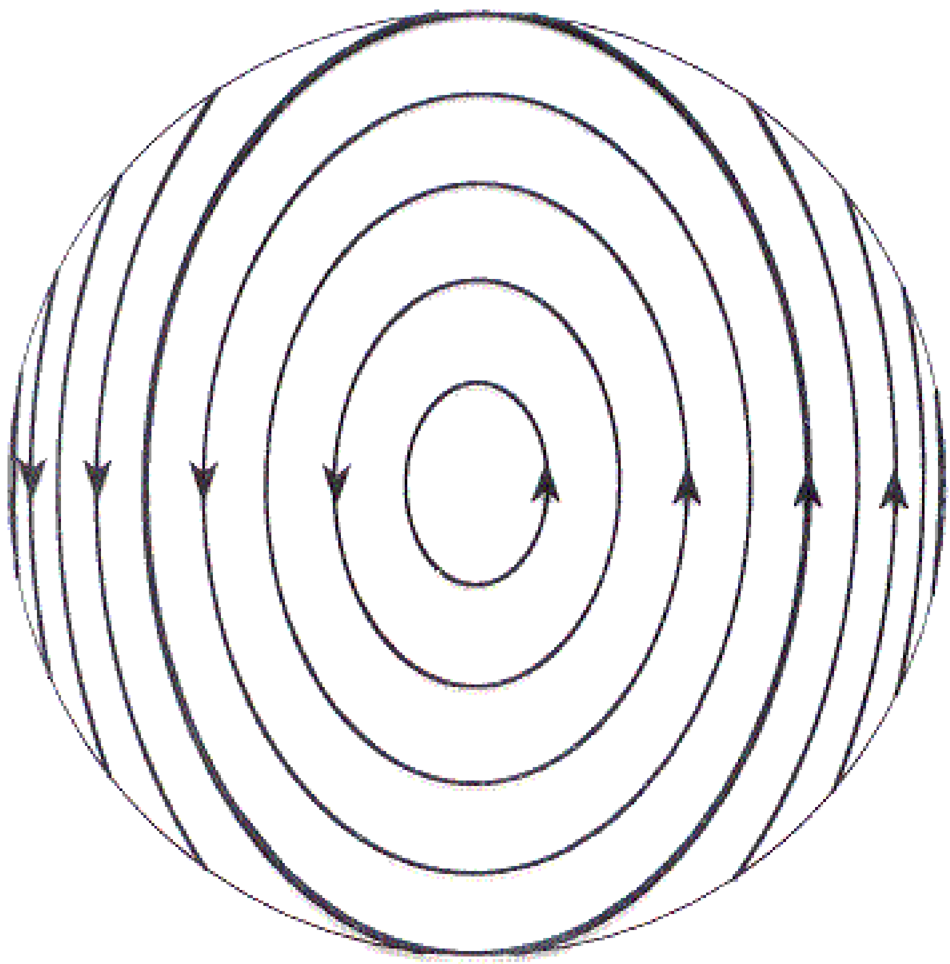
- The flow pattern moves in the co-rotating frame of the star with angular velocity $-(\omega + m\Omega)/m$ ($-\frac{1}{3}\Omega$ for the $m=2$ case) in the opposite direction as the star's rotation.

Flow Pattern for the $m = 2$ r-mode

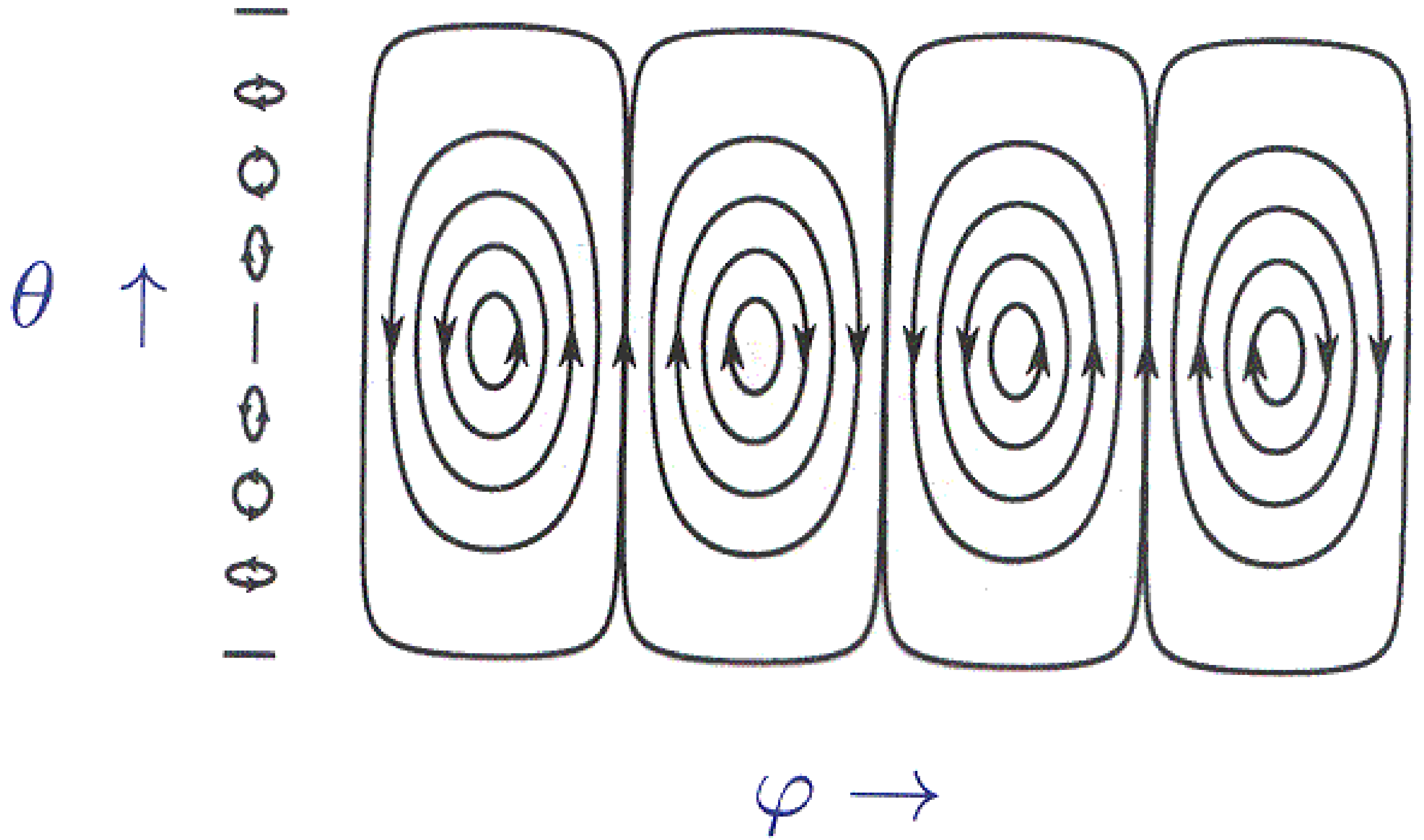
Polar View



Equatorial View



Fluid Motion in the $m = 2$ r-mode



- The flow pattern is shown along with the small elliptical paths (on the left) of individual fluid elements. The flow pattern moves (to the left) past the fluid particles as the mode evolves.

- Andersson, Friedman, and Morsink showed that the r-modes are unstable due to gravitational radiation.

- Evaluate the evolution of the energy of the mode (L[2], Owen, and Morsink):

$$\tilde{E} = \tfrac{1}{2}\int \rho\, \delta\vec{v}^{\,*} \cdot \delta\vec{v}\, d^3x + \mathcal{O}(\Omega^4).$$

$$\frac{d\tilde{E}}{dt} = -\omega(\omega+m\Omega) \sum_{l\geq m} N_l\, \omega^{2l} \left(|\delta D_{lm}|^2 + |\delta J_{lm}|^2\right).$$

- For the r-modes

$$\omega(\omega + m\Omega) = -\frac{2(m-1)(m+2)}{(m+1)^2}\Omega^2 < 0.$$

- Thus $d\tilde{E}/dt > 0$ and gravitational radiation makes *all* r-modes in *all* rotating stars unstable.

- Viscosity also effects the r-mode evolution

$$\frac{d\tilde{E}}{dt} = -\int \left(2\eta\delta\sigma^*_{ab}\delta\sigma^{ab} + \zeta\delta\sigma^*\delta\sigma\right) d^3x$$

$$-\omega(\omega + m\Omega) \sum_{l\geq m} N_l\, \omega^{2l} \left(|\delta D_{lm}|^2 + |\delta J_{lm}|^2\right)$$

- The imaginary part of the frequency of the mode is given by

$$\frac{1}{\tau} = -\frac{1}{2\tilde{E}}\frac{d\tilde{E}}{dt} = \frac{1}{\tau_{GR}} + \frac{1}{\tau_V}.$$

- For the $m = 2$ r-mode

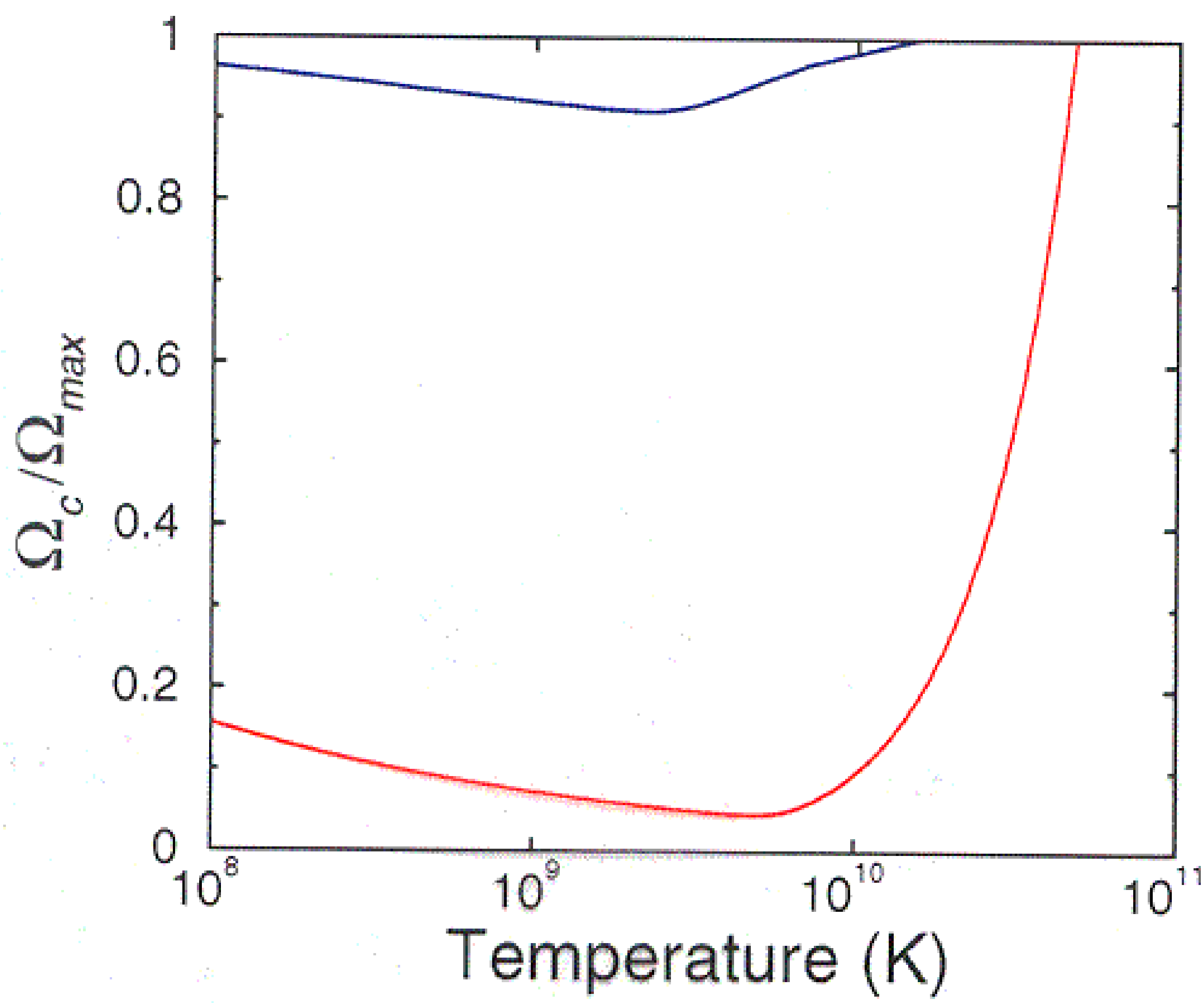
$$\frac{1}{\tau_{GR}} = -\frac{1}{3.3s}\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right)^3$$

$$\frac{1}{\tau_V} = \frac{1}{3\times 10^8 s}\left(\frac{10^9 K}{T}\right)^2 + \frac{1}{2\times 10^{11} s}\left(\frac{T}{10^9 K}\right)^6\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right)$$

- Define the critical angular velocity Ω_c as the point where the sign of the imaginary part of the frequency changes:

$$\frac{1}{\tau(\Omega_c)} = 0$$

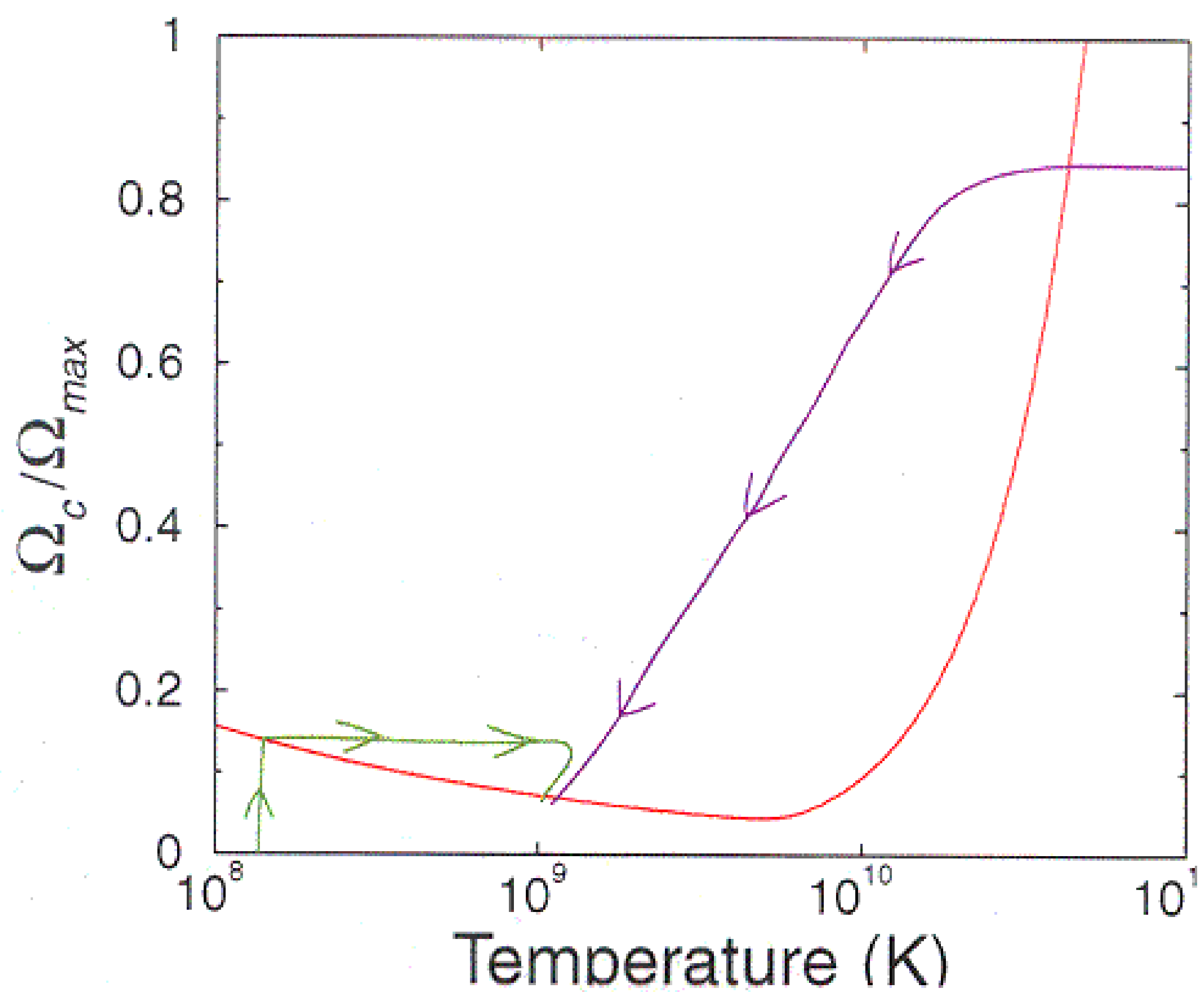
- Rapidly rotating stars $\Omega > \Omega_c$ are unstable even when internal fluid dissipation is included. (The curves represent the r-mode and f-mode stability curves respectively.)



- A rapidly rotating neutron star may become unstable $\Omega > \Omega_c$ in two ways:

1) It may be born hot ($T \approx 10^{11}$K) and rapidly rotating and then cool to become unstable.

2) It may be spun up by accretion until it crosses the instability line and becomes unstable.



Solid Crust Effects

- Below about 10^{10} K a solid crust forms in the outer layers of a neutron star where the density is lower than $\rho_c \approx 1.5 \times 10^{14}$ g/cm^3.

- A viscous boundary layer of thickness d forms in the fluid below the crust. For r-modes

$$d \approx \sqrt{\frac{\nu}{2\Omega}} \approx 5 \text{ cm} \left(\frac{10^8 \text{K}}{T}\right) \left(\frac{\sqrt{\pi G \bar{\rho}}}{\Omega}\right)^{1/2}$$

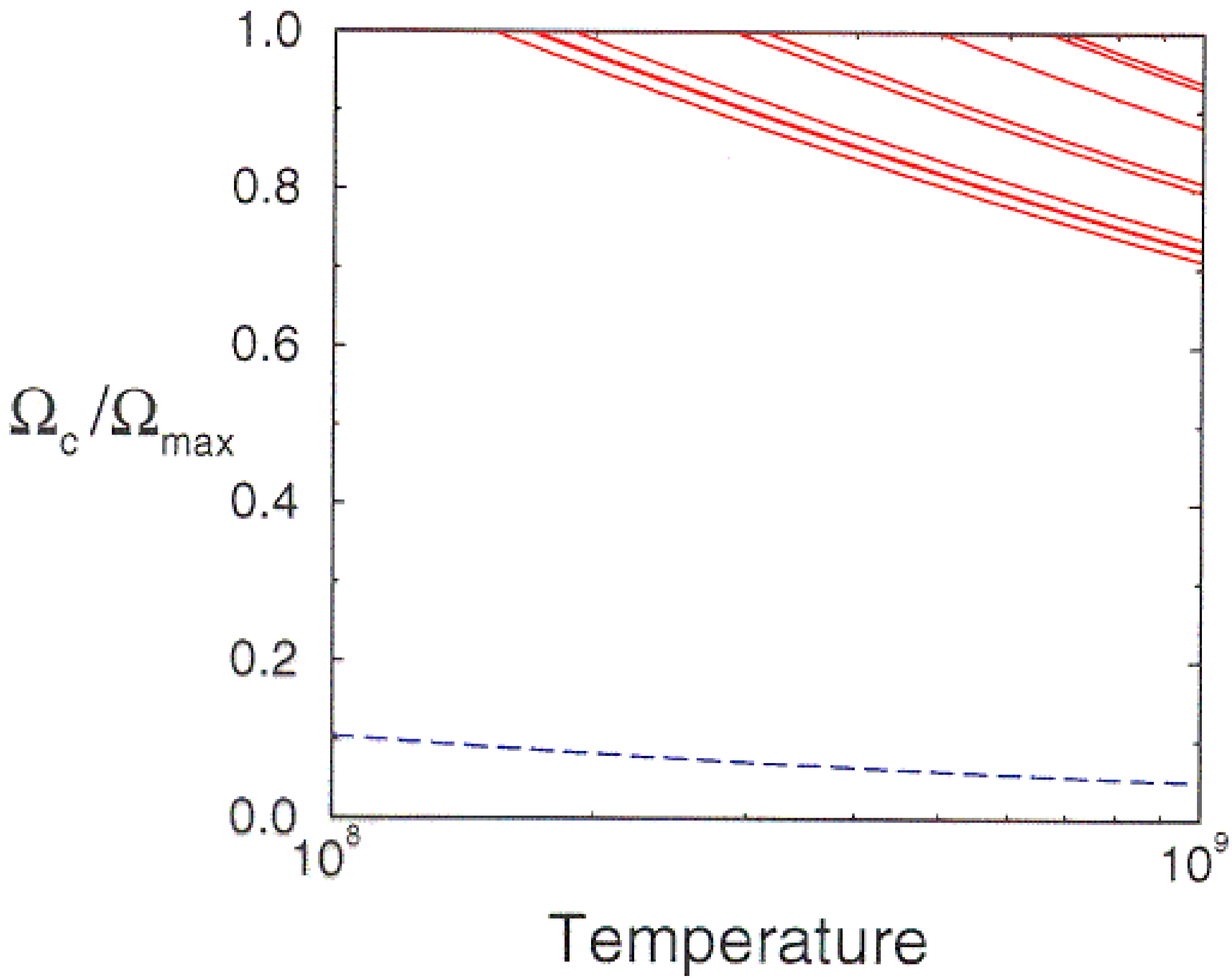
- The shear in the boundary layer is approximately $\nabla v \approx v/d$. This is larger that the usual shear in an r-mode by the factor $R/d \approx 2 \times 10^5$.

- Thus the viscous damping times are shorter than they would have been without the crust by approximately the factor d/R, and so viscosity is much more effective at damping out the r-mode instability once a crust forms.

- The viscous damping time due to shear viscosity in the boundary layer has been evaluated for typical neutron star parameters:

$$\tau_v = \begin{pmatrix} 23\,\text{s} & T < 10^9\text{K} \\ 53\,\text{s} & T > 10^9\text{K} \end{pmatrix} \left(\frac{T}{10^8\text{K}}\right) \left(\frac{\sqrt{\pi G\bar{\rho}}}{\Omega}\right)^{1/2}$$

- The shear viscosity in the boundary layer significantly suppresses the r-mode instability. This allows colder stars $T < 10^9$K to be spun up to large angular velocities.



What about hot young stars?

- An r-mode with initial amplitude $\alpha_o \approx 10^{-6}$ will grow to unit amplitude in a rapidly rotating star in about 10^3s. If the mode amplitude saturates at $\alpha \approx 1$ the star will loose most of its angular momentum to gravitational radiation on a timescale of about 1 year.

- What could prevent the amplitude of the r-mode from growing to values of order unity: hydrodynamic effects (e.g. turbulence), crust formation, magnetic fields ...?

• What about a solid crust? If a crust did form, viscous dissipation in the boundary layer heats the material near the crust.

• If the amplitude of the r-mode exceeds a critical value $\alpha > \alpha_c$ the temperature at the inner edge of the crust will exceed the melting temperature. A detailed calculation gives the estimate

$$\alpha_c \approx 2.8 \times 10^{-3} f(\theta) \left(\frac{T_m}{10^{10}\,\mathrm{K}}\right)^{5/2} \left(\frac{\sqrt{\pi G \bar{\rho}}}{\Omega}\right)^{5/4}$$

where $1 \leq f(\theta) < 2$ except very near the rotation axis of the star.

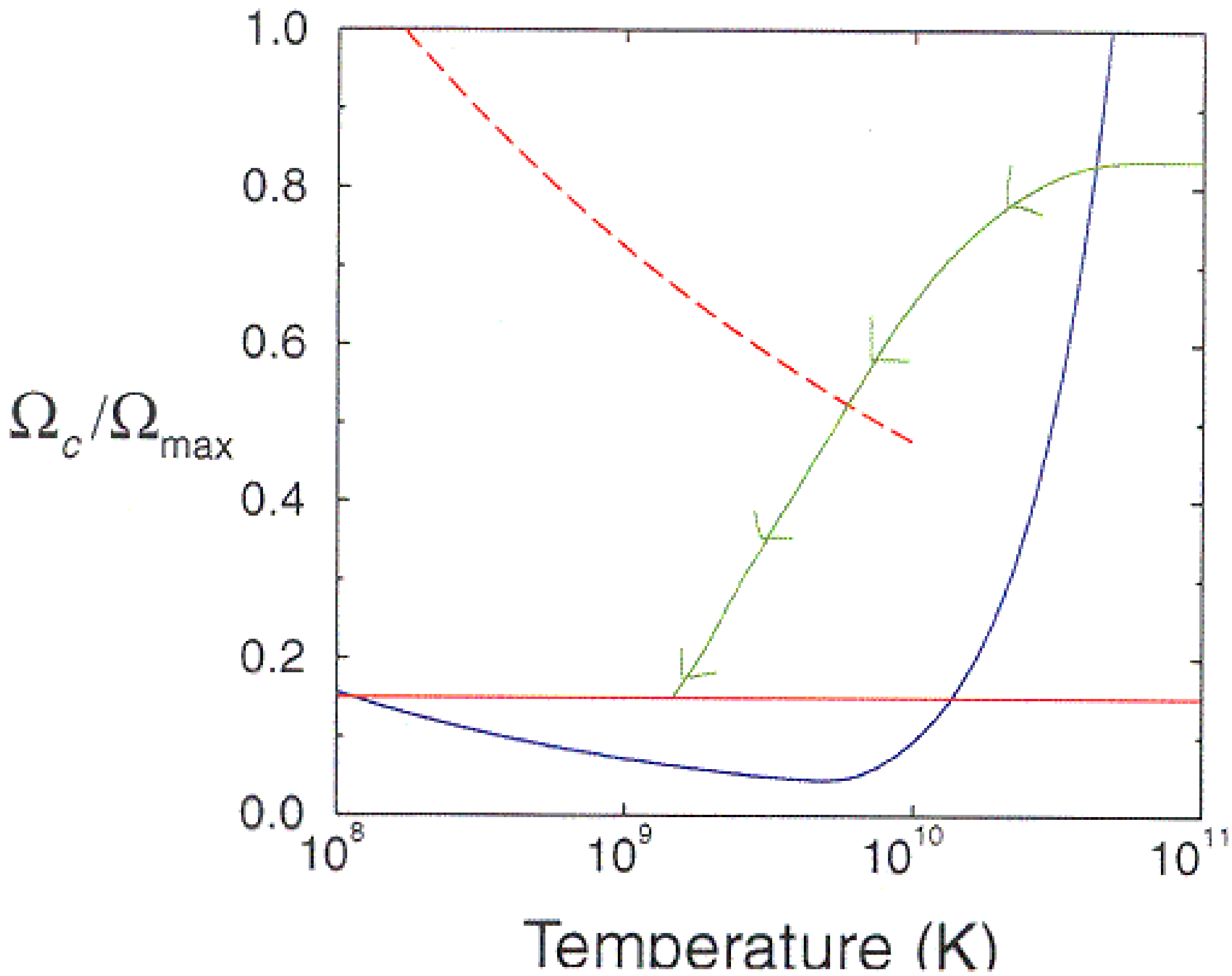
• In a rapidly rotating hot young neutron star the amplitude of the r-mode may grow beyond α_c before the crust has formed.

• If a crust forms viscous, dissipation in the boundary layer would quickly re-melt it. If a crust does not form, neutrio emission quickly lowers the temperature below the melting point.

- What happens? Probably something in between. I expect an "ice flow" (similar to the pack ice on the arctic ocean) to form. Dissipation of the r-mode energy in this flow (dominated by collisions between chunks of ice) will keep the local temperature fixed at the melting temperature.

- The r-mode will continue to have enough energy to maintain the ice flow as long as

$$\Omega > \Omega_c \approx 0.15\alpha^{-1/4}$$



Non-Linear Hydrodynamic Effects

- If the r-mode instability is to remove a significant fraction of the star's angular momentum before the star gets too cold, the amplitude of the mode must grow to order unity: $\alpha \approx 1$.

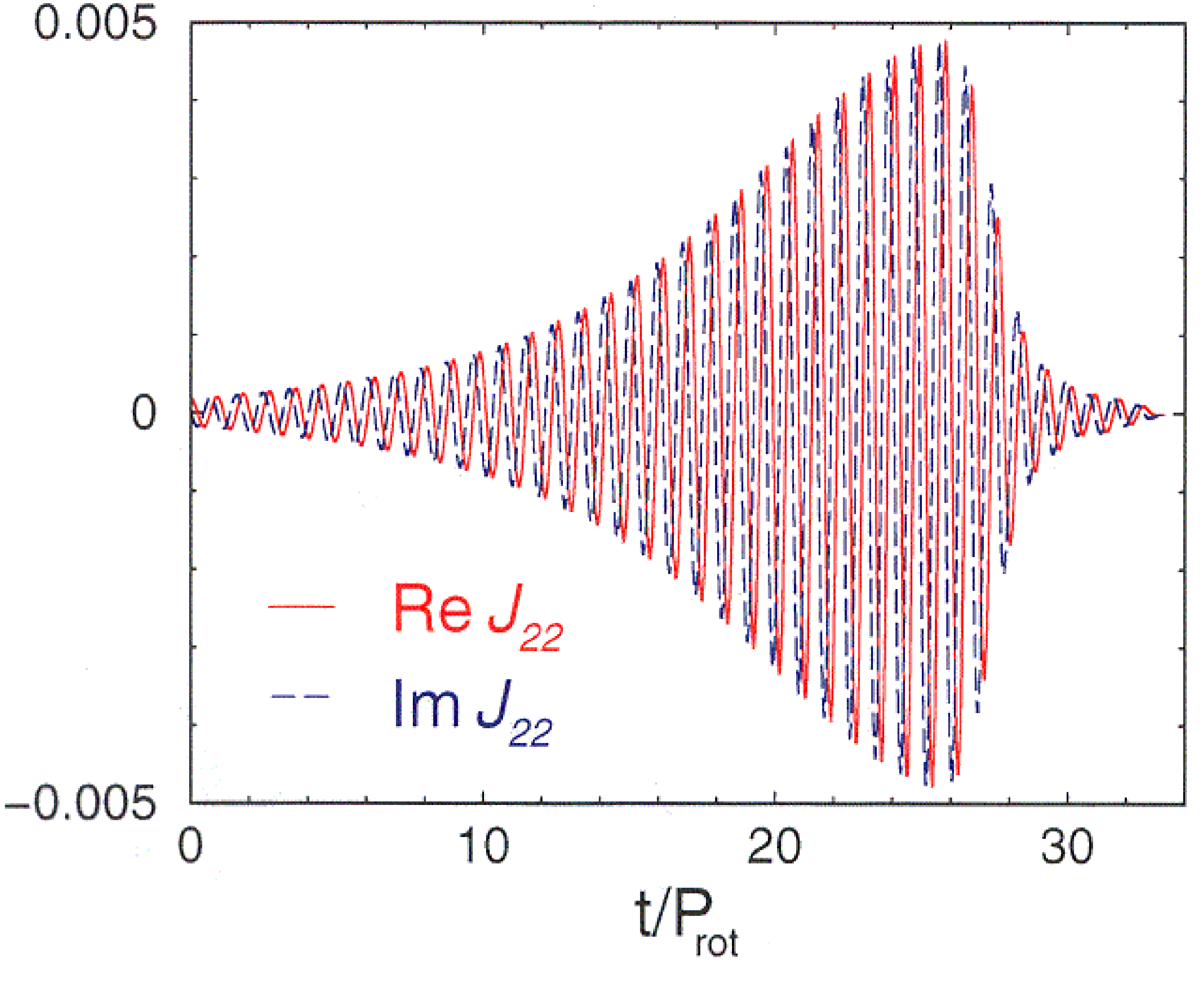
- Do non-linear hydrodynamic forces prevent the mode from reaching such a large amplitude?

- Evolve small-amplitude r-mode initial data on a rapidly rotating ($\Omega \approx 0.95\Omega_{\text{max}}$) neutron star including current-quadrupole gravitational radiation reaction.

- We (Michele Vallisneri, Joel Tohline, & L^2) have evolved such a model on a $64 \times 128 \times 128$ grid using the non-linear 3D Newtonian hydro code developed and tested by Tohline and co-workers over the past decade.

- Monitor the r-mode evolution using J_{22}, the current multipole moment:

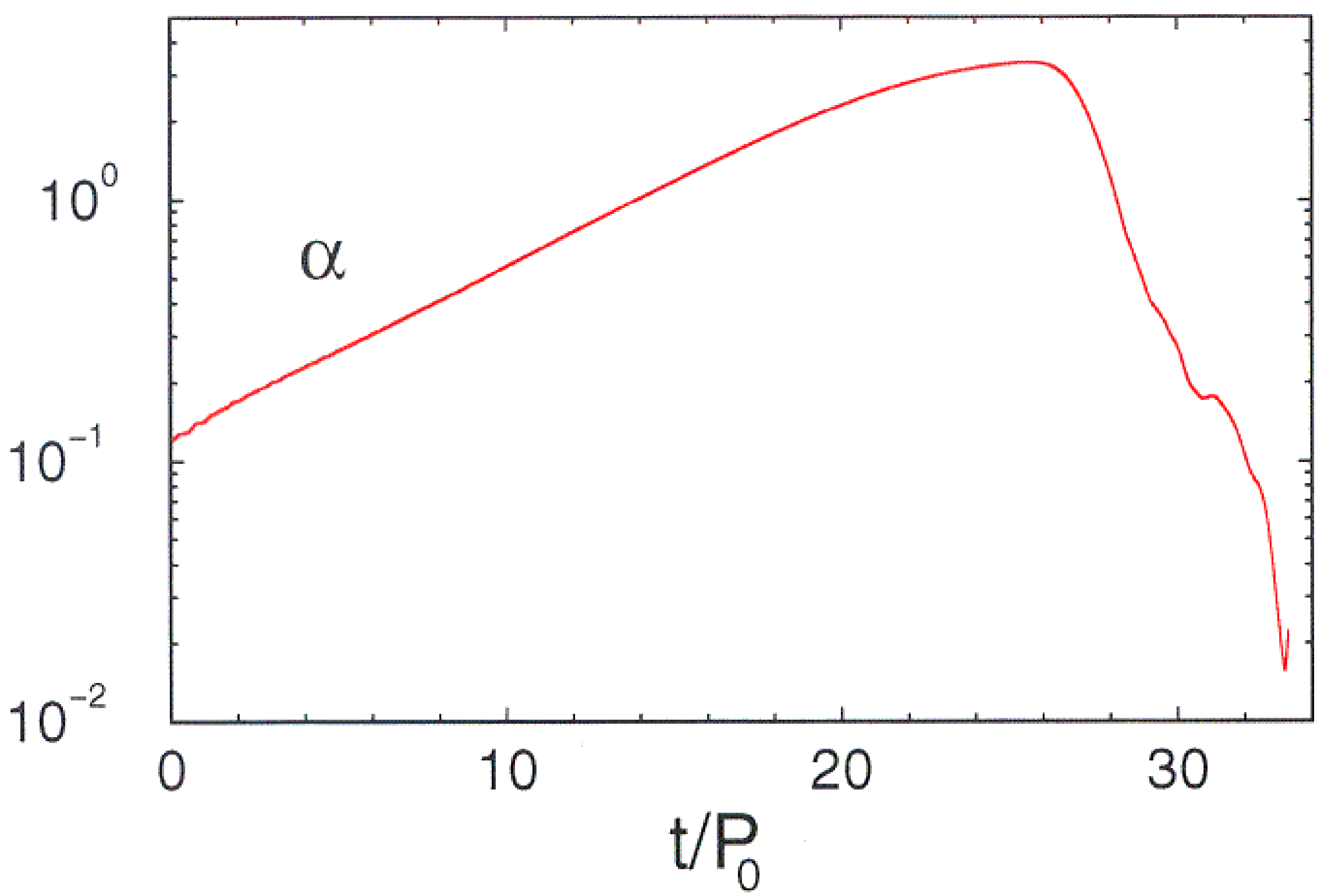
$$J_{lm} = \int \rho r^l \vec{v} \cdot \vec{Y}_{lm}^{B*} \, d^3x$$



- The amplitude of the r-mode

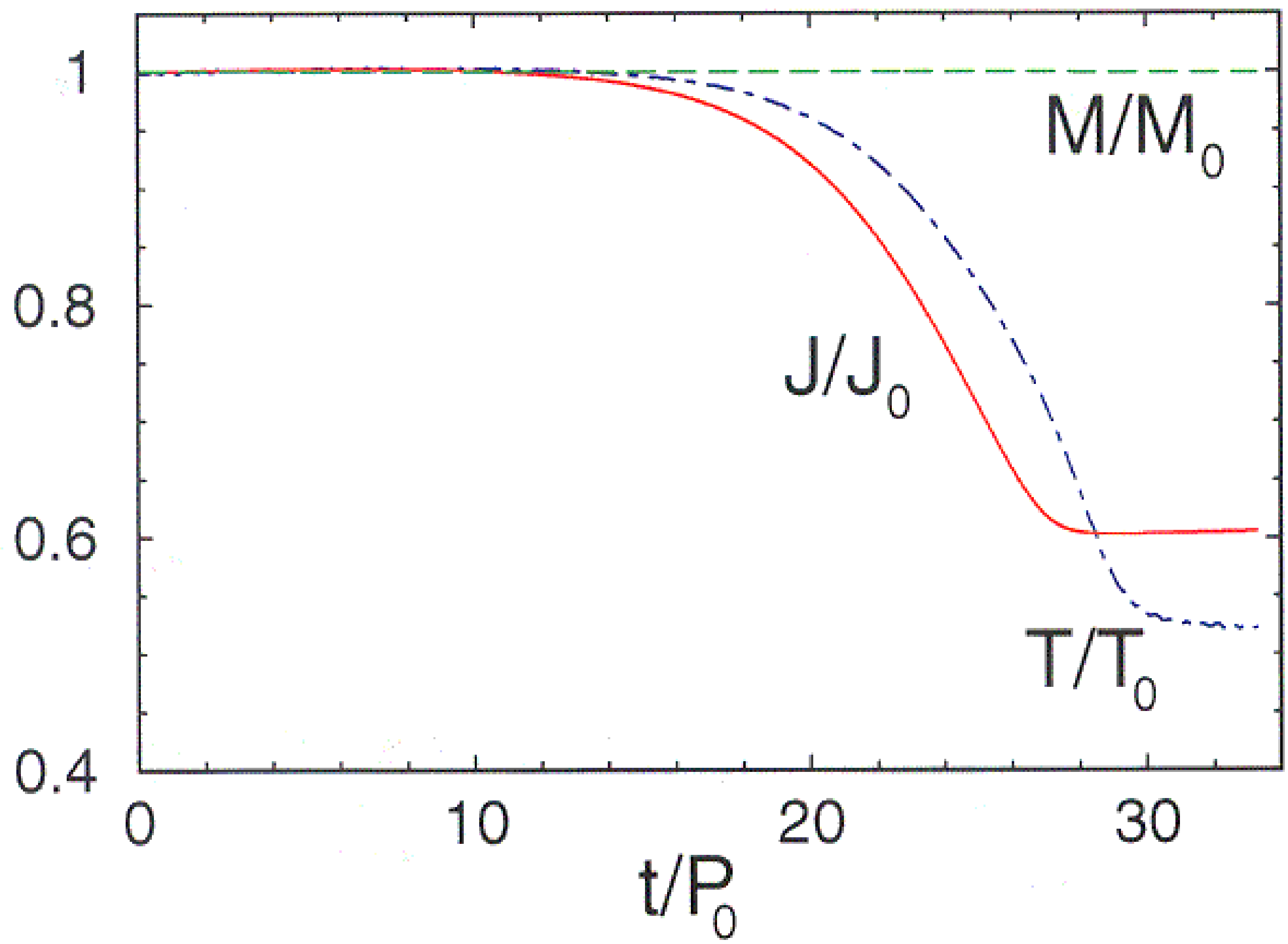
$$\alpha = \frac{2R_o \int \rho r^l \vec{v} \cdot \vec{Y}_{lm}^{B*} \, d^3x}{\Omega_o \int \rho_o r^4 \, d^3x}$$

grows (exponentially at first) due to gravitational radiation reaction. The amplitude reaches the value $\alpha_{max} = 3.35$ and then quickly drops as strong shocks dissipate the energy in the mode.



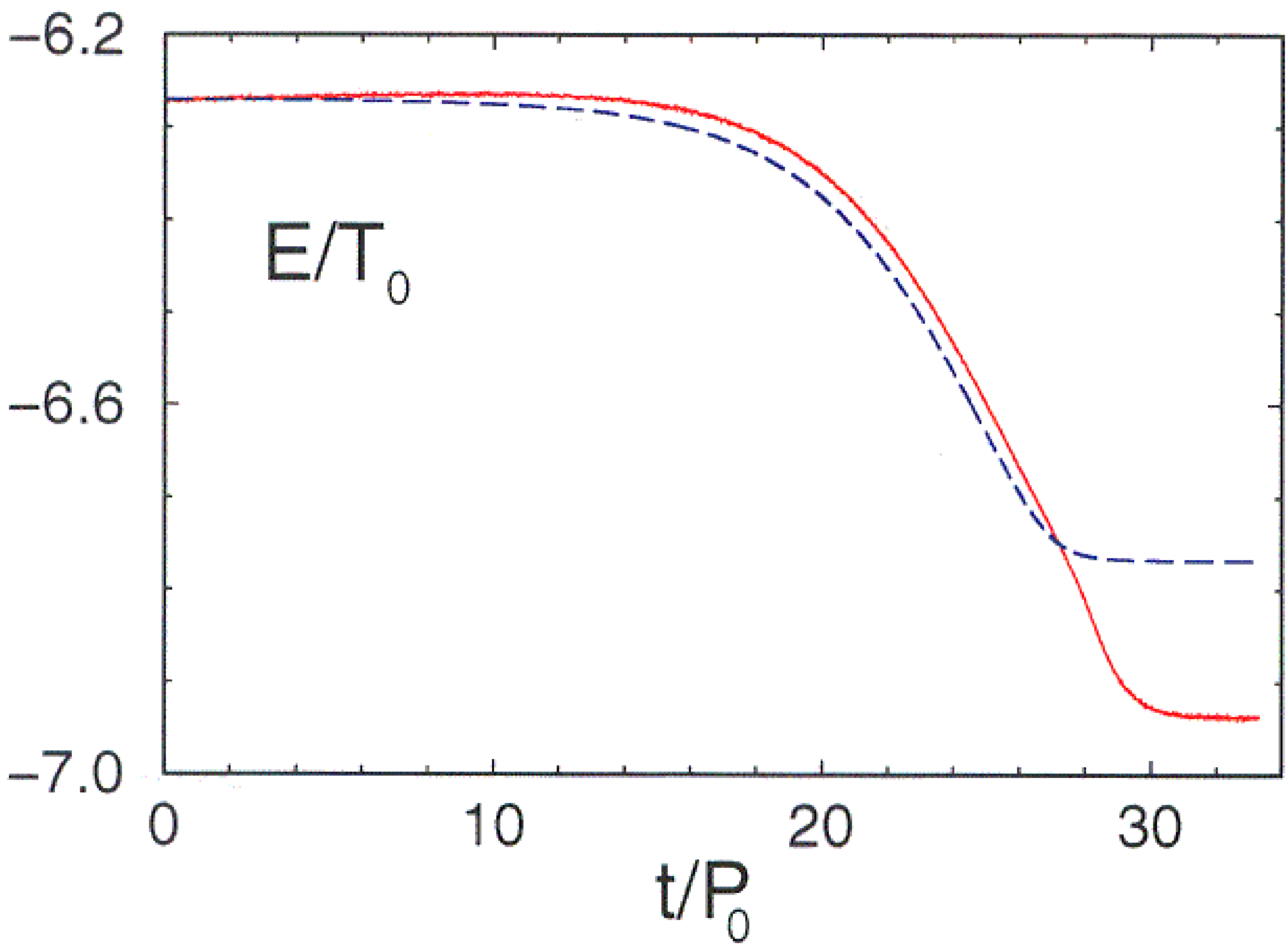
- As the star evolves its total mass M is conserved, while the angular momentum J and rotational kinetic energy T decrease due to the emission of gravitational radiation.

- The star looses about 40% of its initial angular momentum, and 50% of its rotational kinetic energy.

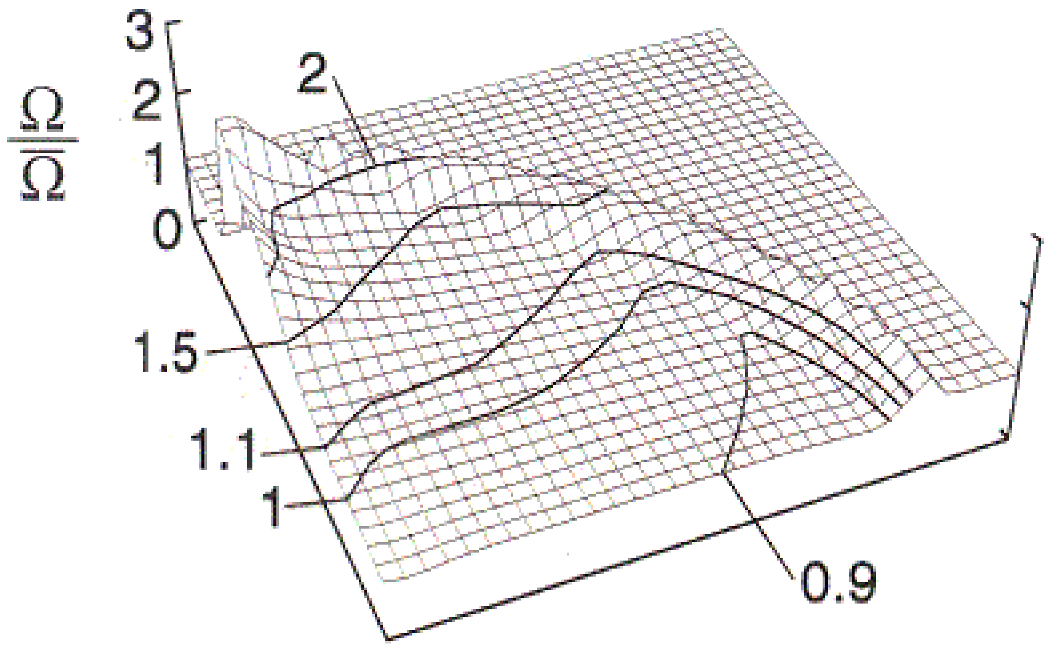


- The evolution of the total energy of the star E (solid curve) tracks the evolution of the energy due to gravitational radiation emission (dashed curve) until shocks form.

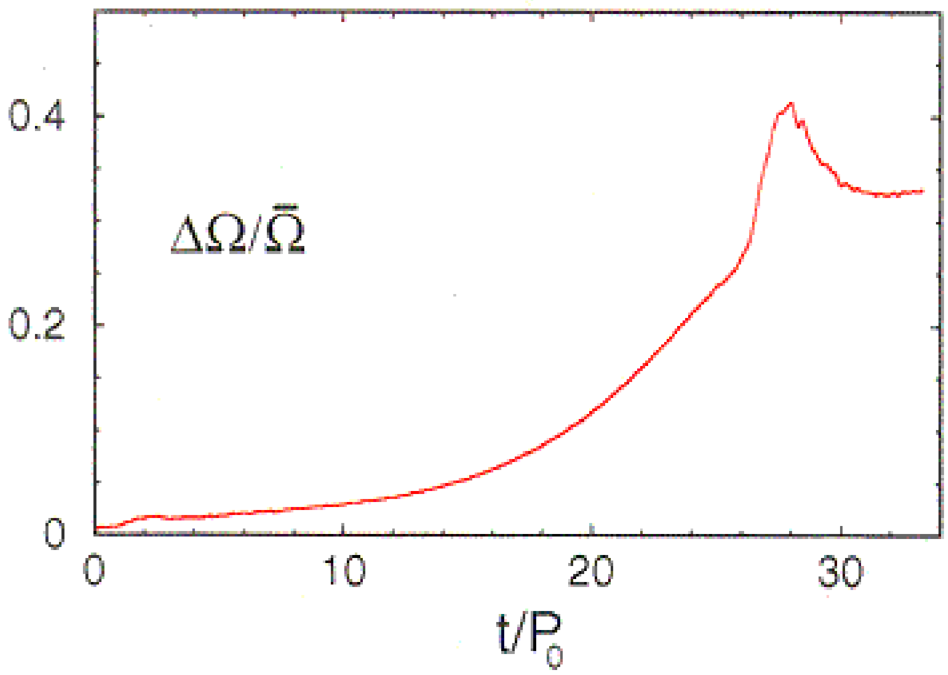
- About 15% of the initial rotational kinetic energy of the star is dissipated in the shocks.



- The non-linear evolution generates differential rotation in the star. The core remains fairly rigidly rotating, but the outer envelope, and especially the polar regions, rotate much more rapidly than the average.



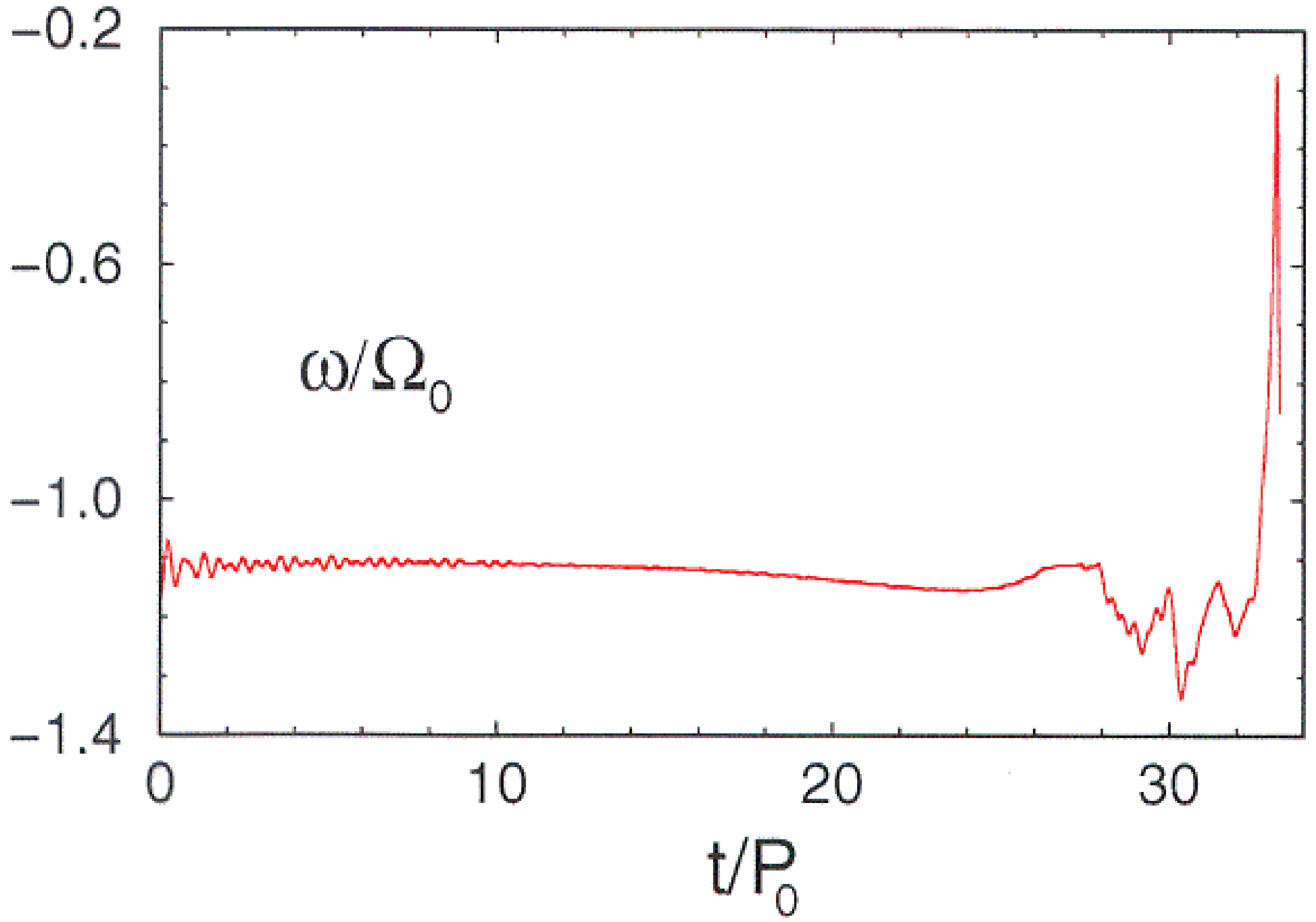
- The average differential rotation $\Delta\Omega$ increases with time.



- The frequency of the mode is defined in terms of J_{22}:

$$\omega = -\frac{1}{|J_{22}|}\left|\frac{dJ_{22}}{dt}\right|$$

- The variation in the frequency of the mode ω is only a few percent during the gravitational radiation dominated phase of the evolution.



Gravitational Radiation from the R-Modes

- The r-mode emits current-quadrupole GR with (angle averaged) amplitude

$$< h^2 > = \frac{1024\pi}{1125}\frac{G^2}{c^{10}}\frac{\omega^4|J_{22}|^2}{D^2}$$

- Using the stationary-phase approximation, the Fourier transform $\tilde{h}$ may be expressed in the form (Blandford):

$$< \tilde{h}^2 > = \frac{G}{10\pi c^3 D^2 f}\frac{dJ}{df}.$$

- The optimal S/N for detecting this type of signal can then be estimated to be (Owen)

$$\left(\frac{S}{N}\right)^2 = 4\int_0^\infty \frac{\tilde{h}^2}{S_h} df \approx \frac{2G\Delta J}{5\pi c^3 f S_h D^2}$$

where ΔJ is the total angular momentum emitted as GR, S_h is the one-sided power spectral density, and D is the distance to the source.

• The analytical estimate of the optimal S/N for our simulation gives

$$\frac{S}{N} \approx \sqrt{\frac{2G\Delta J}{5\pi c^3 f S_h D^2}} \approx 2.7$$

for a source located at $D = 20$ Mpc.

• The determination of $\tilde{h}$ and hence S/N by direct Fourier transform and numerical integration of the simulation data gives the value $S/N \approx 2.5$.