A toroidal solution of the vacuum Einstein field equations*

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This paper presents a new solution to the vacuum Einstein field equations for the static external gravitational field of a toroidal singularity. This solution is unique among known toroidal solutions in that the singularity is locally cylindrically symmetric; near it the spacetime geometry becomes that of an infinite line mass ("Levi-Civita metric").

I. INTRODUCTION

Infinitely long, cylindrically symmetric systems have played a useful role, since 1919, as tools for gaining insight into general relativistic phenomena. For example, much of the pioneering work on gravitational-wave theory dealt with cylindrical systems;[1] and in recent years cylindrical systems have been used as a testing ground for ideas about highly nonspherical gravitational collapse.[2]

A key difficulty with all cylindrical analyses is the fact that spacetime is not asymptotically Minkowskiian far outside a cylindrical system: Just as the Newtonian potential of a cylinder diverges logarithmically at large radii ($\Phi = \text{const} \times \ln r$), so its general relativistic analog, $\Psi \equiv \frac{1}{2}\ln|g_{00}|$, diverges logarithmically. As a result, the physical interpretation of cylindrical spacetimes is often uncertain.

One way to remedy this problem is to deal with systems that are locally cylindrical, but are confined to a finite region of space—e.g., needles (finite cylinders) and thin rings (toruses). Unfortunately, such systems are far more difficult to analyze than are infinitely long cylinders. The purpose of this paper is to present a tool that may be helpful in future analyses of bounded, locally cylindrical systems. That tool is a static, two-parameter solution of the vacuum Einstein field equations representing the external gravitational field of a torus. Unlike other toroidal solutions, very near the ring singularity this one is cylindrically symmetric.

II. THE SOLUTION IN GENERAL

A. The Weyl formalism

In presenting the new solution, we shall use Weyl's formalism[3] for axially symmetric, vacuum solutions of the Einstein field equations. The Weyl formalism is couched in the mathematical language of a flat "background space" with cylindrical coordinates (ρ, z, ϕ) and with metric

$$d\sigma^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2. \tag{1}$$

Two gravitational potentials with axial symmetry reside in the background space: $\psi(\rho, z)$ and $\gamma(\rho, z)$. They satisfy the field equations

$$\psi_{,\rho\rho} + \rho^{-1}\psi_{,\rho} + \psi_{,zz} = 0, \tag{2a}$$

$$\gamma_{,\rho} = \rho(\psi_{,\rho}{}^2 - \psi_{,z}{}^2), \tag{2b}$$

$$\gamma_{,z} = 2\rho\psi_{,\rho}\psi_{,z}, \tag{2c}$$

where commas denote partial derivatives. It is often useful to rewrite these field equations in terms of the gradient operator ∇ and Laplacian ∇^2 of the flat background space (1):

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial\rho} + \mathbf{e}_z \frac{\partial}{\partial z}, \quad \nabla^2 = \rho^{-1}\frac{\partial}{\partial\rho}\frac{\rho\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}, \tag{2a$'$}$$

$$\nabla^2\psi = 0, \quad |\nabla\gamma| = \rho(\nabla\psi)^2, \tag{2b$'$}$$

If $\nabla\psi$ makes an angle θ_0 with the radial $(\mathbf{e}_\rho)$ direction, then $\nabla\gamma$ makes an angle $2\theta_0$ with the radial direction (2c′)

Corresponding to any solution of the field equations (2) or (2′) in the flat background space (1), there exists a static, axially symmetric solution of the vacuum Einstein field equations with the metric

$$ds^2 = -\exp(2\psi)\,dt^2 + \exp[2(\gamma - \psi)]\,(d\rho^2 + dz^2) + \rho^2\exp(-2\psi)\,d\phi^2. \tag{3}$$

Different solutions are obtained by choosing different singular souces for ψ in the background space (point sources, line sources, surface sources). If the sources are confined to a finite region of the background space, then both ψ and γ will approach constants as $(\rho^2 + z^2)^{1/2} \to \infty$; those constants can be chosen zero without loss of generality, and the resulting physical spacetime (3) is asymptotically Minkowskiian.

B. Toroidal solutions that are not locally cylindrical

The easiest way to construct toroidal solutions is to choose, as the source of ψ in the background space, a singularity at $\rho = b$, $z = 0$ (ring singularity around axis of symmetry). The simplest ring singularity is a pure "line monopole," for which[4]

$$\psi = \text{const} \times \ln[(\rho - b)^2 + z^2]^{1/2} \text{ near singularity, i.e., at } [(\rho - b)^2 + z^2]^{1/2} \ll b. \tag{4}$$

Unfortunately, when ψ has this locally cylindrical form, γ and the physical metric are not locally cylindrical near the singularity; Eq. (2c′) forbids it. One cannot remedy this problem by any other choice for the ring source of ψ (any superposition of line multipoles at $\rho = b$, $z = 0$).[5]

This situation is analogous to the case of spherical symmetry: No type of point singularity in the background space (no superposition of point multipoles) can lead to a spherical physical metric; Eq. (2c′) forbids it. To get a spherical metric (the Schwarzschild solution), one must choose as the source of ψ a *"line* mass" on the axis of symmetry, with "mass per unit length" $\frac{1}{2}$ (so $\psi = \frac{1}{2}\ln\rho$ near it), and with finite length $\Delta z = 2M$ = ("Schwarzschild radius").[6]

C. The potentials ψ and γ for the new toroidal solution

It turns out that the background-space source for a locally cylindrical, globally toroidal metric is even more peculiar than that for the Schwarzschild solution. The desired source is best understood by thinking of the background space as filled with an incompressible fluid that undergoes steady-state potential flow with potential ψ and with momentum density $\rho_0\mathbf{v} = \nabla\psi$ (ρ_0, not to be confused with ρ, is the mass density of the fluid). The fluid is created in a line singularity on the axis of symmetry (Fig. 1), and flows outward from there. The singularity has a finite height, $z = 2a$; and it pours out fluid at a constant rate $\mathring{m}$. Once created, the fluid does not freely expand into the background space. Rather, its flow is constrained by two solid disks that are attached to the ends of the source ($z = \pm a$) and that have radii b (Fig. 1).

By the time the flowing fluid gets far from the constraining disks, $r \equiv (\rho^2 + z^2)^{1/2} \gg b$, its flow has become nearly spherical with mass flow rate

$$\mathring{m} = 4\pi r^2 \rho_0 v^r = 4\pi r^2 \psi_{,r} \tag{5}$$

and potentials

$$\psi = -(\mathring{m}/4\pi) r^{-1}, \quad \gamma = O(\mathring{m}^2 r^{-2}). \tag{6}$$

Hence, the physical spacetime metric (3) has the asymptotic form

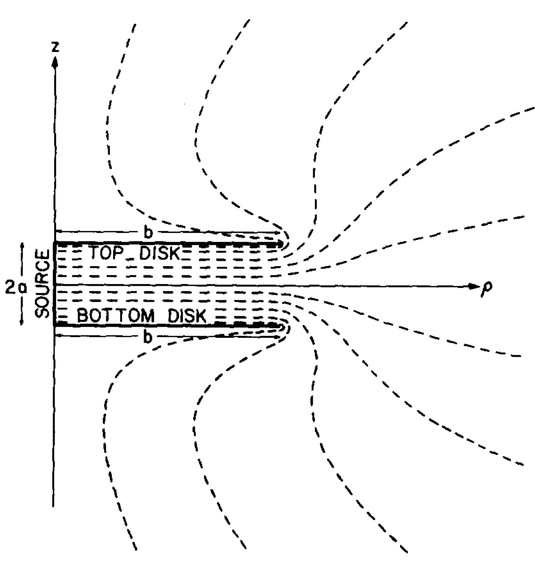

FIG. 1. The flow of fluid in the (fictitious) background space. The flow lines (trajectories of $\nabla\psi$) are shown dashed.

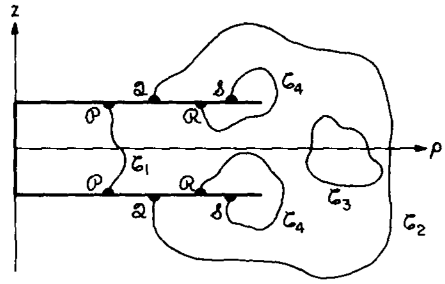

FIG. 2. The toroidal topology of physical spacetime. The events $\mathcal{P}$ and $\mathcal{R}$ are located, in coordinate space, on the inside faces of both the upper disk and the lower disk: $\mathcal{Q}$ and $\mathcal{S}$ are located on both outside faces. The closed curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are topologically linked through the ring singularity; the singularity prevents them from being contracted to a point. The closed curves $\mathcal{C}_3\,\mathcal{C}_4$ do not link the singularity; they can be contracted to a point.

$$ds^2 = -[1-(\mathring{m}/2\pi)r^{-1}]\,dt^2 + [1+(\mathring{m}/2\pi)r^{-1}] \times (d\rho^2 + dz^2 + \rho^2\,d\phi^2), \tag{7}$$

from which we can read off the total mass-energy M of the gravitating system in terms of the mass flow rate $\mathring{m}$ in the (fictitious) background metric:

$$M = \mathring{m}/4\pi. \tag{8}$$

Near the singular source the flow is in the $\mathbf{e}_\rho$ direction (see Fig. 1), with

$$4\pi M = \mathring{m} = (2a)(2\pi\rho)\psi_{,\rho} \tag{9}$$

and thus with

$$\psi = (M/a)\ln\rho + \text{const} \quad \text{at} \quad |z| < a, \quad \rho \ll \max(a,b). \tag{10a}$$

The solution for ψ *can be summarized mathematically as follows*: (i) ψ has the asymptotic form (10a) near the singularity; (ii) ψ satisfies the boundary conditions

$$\psi_{,z} = 0 \quad \text{at} \quad z = \pm a, \quad \text{for } 0 < \rho < b \tag{10b}$$

("fluid flow constrained by disk"); (iii) everywhere ψ satisfies

$$\nabla^2\psi = 0 \tag{10c}$$

("potential flow"); (iv) ψ vanishes at spatial infinity

$$\psi = -M/r \quad \text{as} \quad r = (\rho^2 + z^2)^{1/2} \to \infty. \tag{10d}$$

The corresponding solution for γ can be summarized by: (v) γ satisfies Eqs. (2b, c) everywhere; and (vi) γ vanishes at spatial infinity.

D. Topology of the new solution

The above discussion fixes the metric coefficients of physical spacetime [Eq. (3)] but does not determine the topology. The topology is fixed by two identifications: (i) the outside face of the upper disk consists of the same events as the outside face of the lower disk:

$$\lim_{\epsilon\to 0}(t,\rho,z=a+\epsilon,\phi) \text{ is same event as } \lim_{\epsilon\to 0}(t,\rho,z=-a-\epsilon,\phi) \quad \text{if } 0 \leqslant \rho \leqslant b; \tag{11a}$$

(ii) the inside face of the upper disk consists of the same events as the inside face of the lower disk:

$$\left.\begin{array}{l}\lim_{\epsilon\to 0}(t,\rho,z=a-\epsilon,\phi)\ \text{is same event as}\\ \lim_{\epsilon\to 0}(t,\rho,z=-a+\epsilon,\phi)\end{array}\right\}\ \text{if } 0<\rho\leq b. \tag{11b}$$

These two identifications endow the singularity of physical spacetime with a toroidal topology; see Fig. 2.

E. Local cylindrical symmetry near the singularity

Near the singularity, i.e., for $\rho \ll \max(a,b)$, ψ and γ have the form

$$\psi=(M/a)\ln\rho+\psi_0,$$

$$\gamma=(M/a)^2\ln\rho+\gamma_0,\ \psi_0 \text{ and } \gamma_0 \text{ constant}; \tag{12}$$

cf. Eqs. (10a) and (2b). The corresponding spacetime metric (3) is

$$ds^2=-\exp(2\psi_0)\rho^{2M/a}dt^2 + \exp(-2\psi_0)[\exp(2\gamma_0)\rho^{2(M/a)(M/a-1)}(d\rho^2+dz^2) + \rho^{2-2M/a}d\phi^2]. \tag{13a}$$

In this region of spacetime, z is a periodic coordinate that encircles the singularity

$$-a\leq z\leq +a,\quad z=-a \text{ is same set of events as } z=+a, \tag{13b}$$

and ϕ is a "longitudinal coordinate" stretching along the singularity. Since the metric coefficients depend only on the radial coordinate ρ, the geometry is cylindrically symmetric. In fact, except for topological closure of the ring (periodicity of longitudinal coordinate ϕ), the spacetime geometry (13) is that of an infinitely long, cylindrically symmetric line mass (Levi-Civita's[7] solution of the Einstein field equations).

F. Free parameters in the solution

At first sight there are three free parameters in the solution: M, b, and a. However, for arbitrary choices of M, b, a there exists a singularity at the common edge of the disks ($\rho=b$, $z=\pm a$). One can see this as follows: The field equation (10c) and boundary condition (10b) guarantee that near ($\rho=b$, $z=\pm a$) ψ has the form

$$\psi=A+B\,\bar{r}^{1/2}\cos\bar{\theta}/2, \tag{14a}$$

where $\bar{r}$ and $\bar{\theta}$ are polar coordinates centered on the edge of the disks (Fig. 3):

$$\bar{r}=[(\rho-b)^2+(z-a)^2]^{1/2},\quad \bar{\theta}=\tan^{-1}[(a-z)/(b-\rho)] \quad \text{near } (b,a);$$

$$\bar{r}=[(\rho-b)^2+(z+a)^2]^{1/2},\quad \bar{\theta}=\tan^{-1}[(-a-z)/(b-\rho)]+2\pi \quad \text{near } (b,-a). \tag{15}$$

The form of γ near the edge of the disks, as fixed by Eqs. (14a) and (2b′, c′), is

$$\gamma=C-\tfrac{1}{4}B^2b\ln\bar{r}. \tag{14b}$$

The constants A, B, C are unique functions of M, b, a—functions which one can determine by fully solving Eqs. (10) and (2). Expressions (14) and (15), when inserted into the physical metric (3), yield

$$ds^2=-\exp(2A)[1+2B\bar{r}^{1/2}\cos(\bar{\theta}/2)]dt^2+\exp(-2A)\times[1-2B\bar{r}^{1/2}\cos(\bar{\theta}/2)]\times[\exp(2C)\bar{r}^{-B^2b/2}(d\bar{r}^2+\bar{r}^2d\bar{\theta}^2)+(b-\bar{r}\cos\bar{\theta})^2d\phi^2]. \tag{16}$$

This metric with its square roots and half angles is ugly. However, the coordinate transformation

$$R=\bar{r}^{1/2},\quad \Theta=\bar{\theta}/2 \quad (\text{so } \Theta \text{ runs from 0 to } 2\pi) \tag{17}$$

brings it into the nicer form

$$ds^2=-\exp(2A)[1+2BR\cos\Theta]dt^2+\exp(-2A)[1-2BR\cos\Theta]\times[4\exp(2C)R^{2-B^2b}(dR^2+R^2d\Theta^2)+(b-R^2\cos2\Theta)^2d\phi^2]. \tag{18}$$

The spacetime geometry described by this metric is perfectly well behaved if $B^2=2/b$; otherwise it possesses a physical singularity at $R=\bar{r}=0$—i.e., on the edge of the disks.

Thus, by demanding that spacetime be nonsingular at the common edge of the disks, we impose the constraint

$$[B(M,b,a)]^2=2/b \tag{19}$$

and thereby reduce the number of free parameters from 3 to 2. It is easy to verify that in this case spacetime is completely free of singularities, except for the locally cylindrical ring source at $\rho=0$, $|z|<a$.

III. THE SPECIAL CASE OF A THIN-RING TORUS

We now specialize our solution to the case

$$b\gg a \tag{20}$$

i.e., (radius of constraining disks in background space) $\gg$ (separation between disks). The spacetime geometry in this special case will turn out to be that of a thin-ring torus with (total mass-energy) $=M \ll$ (radius of ring) $=b$; see Sec. IVA, below.

In this special case we shall solve explicitly but approximately for the metric coefficients. The errors in our solution will vanish in the limit $a/b\to 0$. Our solution will have different forms in three different regions (see Fig. 4):

Region I: $[(\rho-b)^2+z^2]^{1/2}\gtrsim(ab)^{1/2}$ always, and $|z|>a$ when $\rho<b$, (21a)

Region II: $[(\rho-b)^2+z^2]^{1/2}\lesssim(ab)^{1/2}$ (21b)

Region III: $|z|<a$, $(b-\rho)\gtrsim(ab)^{1/2}$. (21c)

Note that Regions I and II overlap and Regions II and III overlap.

A. Region I

Region I is the "external region" that lies outside the constraining disks and is bounded away from their edges.

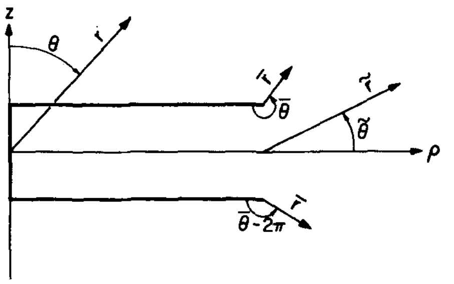

FIG. 3. Various coordinate systems used in the background space. Note that

$$\rho = r\sin\theta = b + \tilde{r}\cos\tilde{\theta} = b - \bar{r}\cos\bar{\theta},$$

$$z = r\cos\theta = \tilde{r}\sin\tilde{\theta} = \begin{cases} a - \bar{r}\sin\bar{\theta} & \text{for } z>0,\ 0<\bar{\theta}<2\pi, \\ -a - \bar{r}\sin\bar{\theta} & \text{for } z<0,\ 2\pi<\bar{\theta}<4\pi. \end{cases}$$

In solving for ψ and γ here, we pretend that the disks are fitted tightly together so that the "fluid" in the background space emerges from a ring singularity at $\rho=b$, $z=0$. This approximation produces

$$\begin{aligned} &(\text{fractional errors in } \psi) \lesssim a/(ab)^{1/2} = (a/b)^{1/2}, \\ &(\text{fractional errors in } \gamma) \lesssim (a/b)^{1/2}[\ln(a/b)^{1/2}]^2. \end{aligned} \tag{22}$$

The solution to the potential-flow equation $\nabla^2\psi=0$ with a ring source at $\rho=b$, $z=0$ and with asymptotic form (10d) is

$$\psi = \frac{-(2/\pi)M}{[(\rho+b)^2+z^2]^{1/2}} K(k), \quad k \equiv \left(\frac{4b\rho}{(\rho+b)^2+z^2}\right)^{1/2}, \tag{23a}$$

$$\approx -\frac{M}{r}\left[1 + O\left(\frac{b^2}{r^2}\right)\right] \quad \text{if } r \equiv (\rho^2+z^2)^{1/2} \gg b \tag{23b}$$

$$\approx \frac{-M}{\pi b}\ln\left(\frac{8b}{\tilde{r}}\right)\left[1 + O\left(\frac{\tilde{r}}{b}\right)\right] \quad \text{if } \tilde{r} \equiv [(\rho-b)^2+z^2]^{1/2} \ll b. \tag{23c}$$

Here $K(k)$ is the complete elliptic function. The corresponding solution to Eq. (2) for γ is[8]

$$\gamma = \frac{M^2k^4}{4\pi^2 b\rho}\left[-K^2 + 4(1-k^2)K\dot{K} + 4k^2(1-k^2)\dot{K}^2\right] + \frac{M^2k^4}{4\pi^2 b^2}\left[-K^2 + 4(1-k^2)K\dot{K} - 4(1-k^2)(2-k^2)\dot{K}^2\right],$$

$$\dot{K} \equiv dK/dk^2, \tag{24a}$$

$$\approx -\tfrac{1}{2}M^2\left[\frac{\sin^2\theta}{r^2} + O\left(\frac{b^2}{r^4}\right)\right] \quad \text{if } r \gg b \tag{24b}$$

$$\approx -\frac{M^2}{\pi^2 b}\left\{\frac{\cos\tilde{\theta}}{\tilde{r}} + O\left[\frac{1}{b}\left(\ln\frac{b}{\tilde{r}}\right)^2\right]\right\} \quad \text{if } \tilde{r} \ll b. \tag{24c}$$

The coordinates (r, θ) used near infinity and $(\tilde{r}, \tilde{\theta})$ used near the ring are shown in Fig. 3. The metric is obtained by inserting expressions (23) and (24) into Eq. (3).

B. Region II

Region II is the "intermediate region" near the common edge of the constraining disks. When solving for ψ and γ in Region II we shall pretend that the edges of the disks in background space are straight rather than curved; i.e., we shall replace the axially-symmetric potential-flow equation $\psi_{,\rho\rho} + \rho^{-1}\psi_{,\rho} + \psi_{,zz} = 0$ by the plane-symmetric potential-flow equation

$$\psi_{,\rho\rho} + \psi_{,zz} = 0; \tag{25a}$$

and we shall set $\rho=b$ in the derivatives of γ:

$$\gamma_{,\rho} = b(\psi_{,\rho}{}^2 - \psi_{,z}{}^2), \quad \gamma_{,z} = 2b\psi_{,\rho}\psi_{,z}. \tag{25b}$$

In doing so we make

$$\begin{aligned} &(\text{fractional errors in } \psi) \lesssim (ab)^{1/2}/b = (a/b)^{1/2}, \\ &(\text{fractional errors in } \gamma) \lesssim (a/b)^{1/2}[\ln(a/b)^{1/2}]^2. \end{aligned} \tag{26}$$

Equation (25a) for ψ must be solved subject to the "flow-around-the-edge-of-the-disks" constraint (10b). The solution can be found by using the conformal transformation

$$\rho + iz = b + (a/\pi)[1 + u + iv + \exp(u+iv)], \quad |v| \le \pi. \tag{27}$$

More specifically, in terms of the function $u(\rho, z)$ the solution is

$$\psi = (M/\pi b)[u - \ln(8\pi b/a)] \tag{28a}$$

$$\approx -\frac{M}{\pi b}\ln\left(\frac{8b}{\tilde{r}}\right)\left[1 + O\left(\frac{a}{\tilde{r}}\right)\right] \quad \text{if } \tilde{r} \gg a \text{ and } |z| > \frac{b-\rho}{|b-\rho|}a \tag{28b}$$

$$\approx -\frac{M}{\pi b}\ln\left(\frac{8\pi b}{a}\right) + \frac{M}{b}\left(\frac{2\bar{r}}{\pi a}\right)^{1/2}\left(\cos\frac{\bar{\theta}}{2}\right)\left[1 + O\left(\frac{\bar{r}^{1/2}}{a^{1/2}}\right)\right] \quad \text{if } \bar{r} \ll a \tag{28c}$$

$$\approx -\frac{M}{a}\left(\frac{b-\rho}{b}\right) - \frac{M}{\pi b}\left[\ln\left(\frac{8\pi b}{a}\right) + 1 + O(\exp[-\pi(b-\rho)/a])\right] \quad \text{if } (b-\rho) \gg a, \ |z| < a. \tag{28d}$$

See Fig. 3 for definitions of the coordinates $\tilde{r}, \tilde{\theta}, \bar{r}, \bar{\theta}$.

By comparing Eq. (28c) with Eq. (14a), we obtain the explicit form of condition (19), which makes the physical geometry nonsingular at the common edge of the disks:

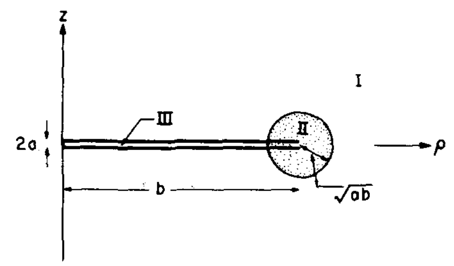

FIG. 4. Three regions, I, II, III, in which three different approximate solutions are valid for the case of a thin-ring torus.

$$B^2 = 2/b \Leftrightarrow M^2 = \pi ab. \tag{29}$$

Henceforth we shall regard M and b as independent variables, and a as the algebraic combination

$$a \equiv M^2/\pi b. \tag{30}$$

The solution for γ can be obtained from Eq. (28) for ψ by integrating Eq. (25b) and imposing the boundary condition (24c) at the outer edge of Region II:

$$\gamma = \frac{M^2}{\pi ab} \operatorname{Re}\left[\ln\left(\frac{\exp(u+iv)}{1+\exp(u+iv)}\right)\right] \tag{31a}$$

$$\approx -\frac{M^2}{\pi^2 ab}\left[\frac{a\cos\tilde{\theta}}{\tilde{r}} + O\left(\frac{a^2}{\tilde{r}^2}\ln\frac{\tilde{r}}{a}\right)\right] \text{ if } \tilde{r} \gg a, |z| > \frac{b-\rho}{|b-\rho|}a \tag{31b}$$

$$\approx \frac{M^2}{2\pi ab}\left[\ln\left(\frac{a}{2\pi\bar{r}}\right) + O\left(\frac{\bar{r}^{1/2}}{a^{1/2}}\right)\right] \quad \text{if } \bar{r} \ll a \tag{31c}$$

$$\approx -\frac{M^2}{a^2}\left\{\frac{b-\rho}{b} + \frac{a}{\pi b} + O\left[\frac{a}{b}\exp\left(-\frac{(b-\rho)\pi}{a}\right)\right]\right\}$$

$$\text{if } b-\rho \gg a, \quad |z| < a. \tag{31d}$$

C. Region III

Region III is the "inner region" between the disks and bounded away from their edges. In solving for ψ and γ here we ignore the existence of the edges, thereby making

$$(\text{fractional errors in } \psi \text{ and } \gamma) \lesssim a/(ba)^{1/2} = (a/b)^{1/2} \tag{32}$$

and thereby obtaining the cylindrically symmetric expressions (12). The constants ψ_0 and γ_0 in those expressions are fixed by matching onto Region II [Eqs. (28d) and (31d)]:

$$\psi = (M/a)\ln(\rho/b) - (M/\pi b)\ln(8\pi b/a), \tag{33}$$

$$\gamma = (M/a)^2 \ln(\rho/b) - M^2/\pi ab. \tag{34}$$

IV. DISCUSSION OF THE SOLUTION

A. The vacuum solution

The asymptotically flat region of spacetime (the region of redshifts small compared to unity and of nearly globally Minkowski geometry) is that region in which $|\psi| \ll 1$ and $|\gamma| \ll 1$. For the thin-ring case (Sec. III, where $M^2 = \pi ab$ and $a \ll b$) all of Region I is asymptotically flat; the strong-field regime begins in Region II. This allows one to perform Newtonian analyses in Region I, using $\psi = \frac{1}{2}\ln|g_{00}|$ as the Newtonian gravitational potential. Straightforward examination of Eq. (23) shows that a Newtonian observer in Region I will regard the source as a thin ring of total mass-energy M and ring radius b.

Notice that the relation $M^2 = \pi ab$ can be rewritten as

$$\frac{2M}{b} = \frac{(\text{"Schwarzschild radius" of ring})}{(\text{"actual radius" of ring})} = \left(\frac{4\pi a}{b}\right)^{1/2}$$

$$\sim (\text{fractional errors in thin-ring formulas}). \tag{35}$$

This says that, for rings of fixed mass M and ever decreasing ring radius b, the "thin-ring approximation" $a \ll b$ breaks down when b becomes of order the Schwarzschild radius $2M$ of the ring. In this limit the general solution of Sec. II remains valid, but the thin-ring formulas of Sec. III fail.

B. The join to an interior solution

The author's PhD thesis[2] develops mathematical tools for the analysis of infinitely long, cylindrically symmetric systems. Those tools should be applicable, with fractional errors $\lesssim O[(a/b)^{1/2}] = O[M/b]$, in Region III of our thin-ring toroidal solution. One tool of particular interest is the following theorem, which can be inferred from Sec. 8-M of the author's thesis:

Consider an infinitely long, nonsingular material cylinder which is momentarily static and which has, as its external gravitational field, the Levi-Civita line-mass metric with "spacetime character" $D^{(+)}$.[9,10] Demand that the cylinder have nonnegative energy density T^{00} on its hypersurface of time symmetry. Then at the surface of the cylinder (point where $T^{\alpha\beta} \to 0$) the "C-energy" scalar U must be positive.[10,11]

In the $D^{(+)}$ Levi-Civita metric, U is $-\infty$ at the singularity and increases monotonically as one moves radially outward. At some radius ρ_c, U becomes zero; and thereafter it continues to increase, approaching $+\frac{1}{8}$ as $\rho \to \infty$. The above theorem says that *any material cylinder with $T^{00} > 0$, which generates the $D^{(+)}$ Levi-Civita metric, must have its surface outside the "critical radius" ρ_c at which $U = 0$.*

Region III of the thin-ring toroidal solution is endowed with a Levi-Civita metric of character $D^{(+)}$. The C-energy scalar at radius ρ can be calculated by combining that metric [Eqs. (3), (33), (34)] with Eq. (7.8) of the author's thesis[10]; the result is

$$U = \tfrac{1}{8}[1 - (b/\rho)^2]. \tag{36}$$

Thus, the critical radius is

$$\rho_c = b. \tag{37}$$

But this radius lies outside Region III—i.e., it is so large that the line element is already showing noticeable deviations from that of Levi-Civita! Thus, one is forced to conclude that *any nonsingular, momentarily (or permanently) static torus which generates the thin-ring metric and which has nonnegative energy density must have its surface outside Region III—i.e., in Region II or Region I.*

This surprising (and, to me, unhappy) result is intimately tied to the fact that the thin-ring toroidal metric of this paper has only two independent parameters. Since the *general* Levi-Civita solution has two free parameters ("mass parameter" and "canonical radius"),[12] one might hope to construct a locally cylindrical, globally toroidal vacuum metric with three independent parameters—two characterizing the Levi-Civita singularity and one characterizing the radius of the ring. By adjusting one of the singularity parameters appropriately, one would then be able to build interior solutions with given M and b and with arbitrarily small surface radii. However, such solutions will not be possible unless one succeeds in adding a new free parameter to the two-parameter vacuum metric of this paper. I have tried, and failed.

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[11]For the concept of "*C*-energy" see Chap. 7 of the author's thesis
[12]See, e.g., Chap. 8 of the author's thesis.