Reading: See the on-line syllabus for lecture-by-lecture readings.

Collaboration policy: See the on-line collaboration policy.

Homework Problems:

1. **The Cosmic Neutrino background:** At temperatures high enough that all species are ultrarelativistic, and if all species have 0 chemical potential, by integrating over an appropriate thermal distribution,

   a) show that the energy densities of photons, neutrinos and electrons are

   \[ \rho_\gamma = aT^4, \quad \rho_{\nu_i} = \rho_{\bar{\nu}_i} = \frac{7}{16} aT^4, \quad \rho_{e^+} = \rho_{e^-} = \frac{7}{8} aT^4, \]

   where \( a = \frac{\pi^2}{15} \frac{k^4}{c^3\hbar^3} \), and \( i = e, \mu, \tau \) (and no more, according to SLAC and LEP \( Z^0 \) width!).

   Also

   b) show that in physical volume \( V \), the entropies are

   \[ S_i = \frac{4}{3} \rho_i \frac{V}{T}. \]

   c) Use these results to show that neutrinos today form a thermal background like the cosmic microwave background, but with \( T_\nu = (4/11)^{1/3} T_{CMB} \). When many classic cosmology texts were written, neutrinos were believed to be massless. Why does the fact that neutrinos are now known to have masses of order \( 10^{-2}\text{eV} \) not affect this result?

   d) Assuming neutrinos are massless, estimate the number of cosmic neutrinos passing through your body each second. How would this be changed if \( m_\nu = 1\text{eV} \) or \( 10^{-2}\text{eV} \)?

   e) Has a cosmic background neutrino ever interacted with a nucleus in your body? In anyone’s?

2. **WIMP dark matter:**

   a) Show that the number density today of stable relics of a particle species \( X \) of mass \( m_X \) which falls out of equilibrium when it is nonrelativistic (this is called “cold dark matter”) is proportional to \( m_X^{-1} \sigma_a^{-1} \), where \( \sigma_a \) is the annihilation cross-section (at the energies characteristic of the freeze-out time).

   b) Thus show that their contribution to \( \Omega_0 \) depends only on \( \sigma_a \), and show that (for chemical potential \( \mu_X = 0 \))

   \[ \Omega_X \approx \frac{7 \times 10^{-27} \text{cm}^3\text{s}^{-1}}{\langle \sigma_a v \rangle} \times (\text{slowly varying logarithms}) \].
c) Before precision electroweak experiments showed that there is room only for 3 neutrino types, it used to be popular to let $X$ be a fourth neutrino species. Show that this would contribute significantly to $\Omega$ only if $m_X \sim 1$ GeV. (Hints for those with a weak particle-physics background: with $\hbar = c = 1$, the weak coupling constant is $G_F = 1.2 \times 10^{-5}$ GeV$^{-2}$, $1$ GeV$^{-2} = 0.4 \times 10^{-27}$ cm$^2$, and $\sigma_a \sim G_F^2 m_X^2$; those able to do so may justify and perhaps improve on the expression for $\sigma_a$.) [Note for information only: It is today popular to let $X$ be the lightest supersymmetric particle, perhaps a neutralino (a linear combination of the supersymmetric partners to the photon, $Z^0$ and/or Higgs boson), which can annihilate into all the usual particles we know about, plus Higgs bosons. Unfortunately, the cross-sections for these depend on many unknown parameters, though experimental limits suggest that these could give annihilation rates and thus dark matter densities of the right values.]

3. Thomson Scattering Optical Depth

In the standard cosmological scenario, all the electrons and protons in the Universe combine to form hydrogen at a redshift $z \sim 1000$– the epoch of recombination. However, at some redshift $z_{\text{reion}}$, stars begin to form and emit radiation that ionizes all the hydrogen in the Universe. If so, then cosmic microwave background (CMB) photons may Thomson scatter from the free electrons en route from the surface of last scattering. Calculate the optical depth $\tau_{\text{reion}}$ for Thomson scattering of CMB photons as a function of the reionization redshift $z_{\text{reion}}$ for $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. Derive an analytic approximation for the redshift $z_{\text{reion}} > \Omega_m^{-1}$. Write your answer in terms of the baryon density $\Omega_b h^2 = 0.023$ (where $h = H_0/100$ km $s^{-1}$ Mpc$^{-1}$) and in terms of the helium mass fraction $Y \simeq 0.23$. At what $z_{\text{reion}}$ does $\tau_{\text{reion}} = 1$?